GENERALIZED LOCAL COHOMOLOGY AND THE CANONICAL ELEMENT CONJECTURE

BY

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Abstract

Let $(A, \mathfrak{m})$ be a local ring of dimension n. The Canonical Element Conjecture asserts that the following canonical map

$$\operatorname{Ext}^n_A(A/\mathfrak{m}, \operatorname{Syz}_n(A/\mathfrak{m})) \to \mathrm{H}^n_{\mathfrak{m}}(\operatorname{Syz}_n(A/\mathfrak{m}))$$

is nonzero. The conjecture has been answered in the affirmative when A contains a field, when the dimension of the ring is 3 or less, and in several other special cases. In [8], Dutta shows that if the canonical map

$$\operatorname{Ext}^i_A(A/\mathfrak{m}, \operatorname{Syz}_i(A/\mathfrak{m})) \to \mathrm{H}^i_{\mathfrak{m}}(\operatorname{Syz}_i(A/\mathfrak{m}))$$

is nonzero, then for $0 \leqslant j \leqslant i$, the canonical map

$$\operatorname{Ext}^j_A(A/\mathfrak{m}, \operatorname{Syz}_j(A/\mathfrak{m})) \to \mathrm{H}^j_{\mathfrak{m}}(\operatorname{Syz}_j(A/\mathfrak{m}))$$

is nonzero. We utilize generalized local cohomology and study the following map:

$$\operatorname{Ext}^i_A(M/\mathfrak{m}M, \operatorname{Syz}_i(M/\mathfrak{m}M)) \to \mathrm{H}^i_{\mathfrak{m}}(M, \operatorname{Syz}_i(M/\mathfrak{m}M))$$

We prove that if the Canonical Element Conjecture is true, then the above map is nonzero for all nonzero finitely generated A-modules M when $0 \leqslant i \leqslant n$. Moreover, we prove that given any nonregular local ring $(A, \mathfrak{m})$ and any $i > 0$, there exists infinitely many finitely generated A-modules M such that the canonical map

$$\operatorname{Ext}^i_A(M/\mathfrak{m}M, \operatorname{Syz}_i(M/\mathfrak{m}M)) \to \mathrm{H}^i_{\mathfrak{m}}(M, \operatorname{Syz}_i(M/\mathfrak{m}M))$$

is nonzero. This result holds irrespective of the validity of the Canonical Element Conjecture.

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1 Introduction

The Canonical Element Conjecture asserts the following:

Conjecture (Hochster [21]). *Given a local ring $(A, \mathfrak{m})$ of dimension n, the canonical map*

$$\operatorname{Ext}^n_A(A/\mathfrak{m}, S_n) \to \mathrm{H}^n_{\mathfrak{m}}(S_n)$$

is nonzero, where $S_n = \operatorname{Syz}_n(A/\mathfrak{m})$.

The canonical element, from which the conjecture draws its name, is the image of the identity map on S_n, denoted by $\mathbb{1}_{S_n}$, in $\mathrm{H}^n_{\mathfrak{m}}(S_n)$. The conjecture asserts that this element is nonzero. For several other equivalent forms of this conjecture, see Section 2.2.

In [5], Dutta discovered a beautiful interpretation of the Canonical Element Conjecture: Let $(A, \mathfrak{m}, k)$ be local ring of dimension n and consider a minimal free resolution of k:

$$\cdots \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow k \longrightarrow 0$$

Let $S_i = \operatorname{Syz}_i(k)$ and break this resolution into short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & S_n \arrow[r] & F_{n-1} \arrow[r] & S_{n-1} \arrow[r] & 0 \\
0 \arrow[r] & S_{n-1} \arrow[r] & F_{n-2} \arrow[r] & S_{n-2} \arrow[r] & 0 \\
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & S_1 \arrow[r] & F_0 \arrow[r] & k \arrow[r] & 0
\end{tikzcd}$$

Apply the functors $\operatorname{Hom}_A(k, -)$ and $\mathrm{H}^0_{\mathfrak{m}}(-)$ to each of the short exact sequences above. Looking at the connecting homomorphisms of the long exact sequences of the corresponding derived functors, one obtains the following commutative diagram, where $n = \dim(A)$ and $S_i = \operatorname{Syz}_i(A/\mathfrak{m})$:

$$\begin{tikzcd}
k \arrow[r, "\delta_0"] \arrow[d, equal, "\vartheta_0"] & \operatorname{Ext}^1_A(k, S_1) \arrow[r, "\delta_1"] \arrow[d, "\vartheta_1"] & \cdots \arrow[r, "\delta_{n-2}"] & \operatorname{Ext}^{n-1}_A(k, S_{n-1}) \arrow[r, "\delta_{n-1}"] \arrow[d, "\vartheta_{n-1}"] & \operatorname{Ext}^n_A(k, S_n) \arrow[d, "\vartheta_n"] \\
k \arrow[r, "\overline{\delta}_0"] & \mathrm{H}^1_{\mathfrak{m}}(S_1) \arrow[r, "\overline{\delta}_1"] & \cdots \arrow[r, "\overline{\delta}_{n-2}"] & \mathrm{H}^{n-1}_{\mathfrak{m}}(S_{n-1}) \arrow[r, "\overline{\delta}_{n-1}"] & \mathrm{H}^n_{\mathfrak{m}}(S_n)
\end{tikzcd} \tag{1.1}$$

The Canonical Element Conjecture states that the map ϑ_n is nonzero. The canonical element as mentioned above is in fact the image of $1 \in k$ pushed through the diagram above. Moreover, one sees that if ϑ_n is nonzero, then ϑ_i is nonzero for all $0 \leqslant i \leqslant n$. In [8, Theorem 3.2], Dutta showed that ϑ_i is nonzero for $0 \leqslant i \leqslant n-1$.

Due to the vanishing of local cohomology modules above the dimension of the ring, the diagram ends in degree n. Instead of considering local cohomology modules, generalize the idea of local cohomology as follows:

Definition. If $(A, \mathfrak{m})$ is a local ring with finitely generated A-modules M and S, the ith ***M*-local cohomology** of S with respect to $\mathfrak{m}$ is defined as:

$$\mathrm{H}^i_{\mathfrak{m}}(M, S) := \varinjlim_t \mathrm{Ext}^i_A(M/\mathfrak{m}^t M, S).$$

These modules were first studied by Grothendieck in [15, Exposé VI]. They were also studied by Herzog in his thesis [17]. Recently, these modules have attracted the interest of others as well. From the work of Suzuki [28] and Herzog and Zamani [18], one finds a result of critical importance: If an A-module M has infinite projective dimension, then $\mathrm{H}^i_{\mathfrak{m}}(M, S)$ might not vanish for $i > \dim(A)$. Thus given a local ring $(A, \mathfrak{m})$ and a finitely generated nonzero A-module M, we may construct a commutative diagram similar to (1.1) above:

$$\begin{tikzcd}
E^0 \arrow[r, "\delta_0"] \arrow[d, "\vartheta_0"] & E^1 \arrow[r, "\delta_1"] \arrow[d, "\vartheta_1"] & \cdots \arrow[r, "\delta_{i-2}"] & E^{i-1} \arrow[r, "\delta_{i-1}"] \arrow[d, "\vartheta_{i-1}"] & E^i \arrow[r, "\delta_i"] \arrow[d, "\vartheta_i"] & \cdots \\
H^0 \arrow[r, "\overline{\delta}_0"] & H^1 \arrow[r, "\overline{\delta}_1"] & \cdots \arrow[r, "\overline{\delta}_{i-2}"] & H^{i-1} \arrow[r, "\overline{\delta}_{i-1}"] & H^i \arrow[r, "\overline{\delta}_i"] & \cdots
\end{tikzcd}$$

Here $E^i = \mathrm{Ext}^i_A(M/\mathfrak{m}M, S_i)$, $H^i = \mathrm{H}^i_{\mathfrak{m}}(M, S_i)$, $S_i = \mathrm{Syz}_i(M/\mathfrak{m}M)$, and the vertical maps are from the definition of $\mathrm{H}^i_{\mathfrak{m}}(M, -)$. Since E^0 is an $A/\mathfrak{m}$ vector space and its generators are preserved by the δ_i's, the commutativity of the diagram shows that if ϑ_i is nonzero, then ϑ_j is nonzero for $0 \leqslant j \leqslant i$. Hence, we choose to study the following canonical maps

$$\mathrm{Ext}^i_A(M/\mathfrak{m}M, S_i) \to \mathrm{H}^i_{\mathfrak{m}}(M, S_i),$$

where M is some finitely generated A-module of infinite projective dimension. Our main goal is to study the nonvanishing criteria for this map, for all $i \geqslant 0$, in the light of the Canonical Element Conjecture.

In Section 3.1 of this thesis, we start to investigate the map mentioned above. In particular, we prove the following proposition:

Proposition 4.1. *Let $(A, \mathfrak{m})$ be a local ring, then*

$$\mathrm{Ext}^i_A(A/\mathfrak{m}, \mathrm{Syz}_i(A/\mathfrak{m})) \to \mathrm{H}^i_{\mathfrak{m}}(\mathrm{Syz}_i(A/\mathfrak{m})) \qquad \textit{is nonzero}$$

if and only if for all nonzero finitely generated A-modules M

$$\mathrm{Ext}^i_A(M/\mathfrak{m}M, \mathrm{Syz}_i(A/\mathfrak{m})) \to \mathrm{H}^i_{\mathfrak{m}}(M, \mathrm{Syz}_i(A/\mathfrak{m})) \qquad \textit{is nonzero.}$$

When $n = \dim(A)$, we see that the validity of the Canonical Element Conjecture and the above corollary imply that the map

$$\mathrm{Ext}^n_A(M/\mathfrak{m}M, S_n) \to \mathrm{H}^n_{\mathfrak{m}}(M, S_n)$$

is nonzero for all nonzero finitely generated A-modules M. From this result and the work of Dutta in [8, Theorem 3.2] and [11, Theorem 1.2], we obtain the following theorem:

Theorem 4.3. *Let $(A, \mathfrak{m})$ be a local ring. For all nonzero finitely generated A-modules M the canonical map*

$$\mathrm{Ext}^i_A(M/\mathfrak{m}M, S_i) \to \mathrm{H}^i_{\mathfrak{m}}(M, S_i)$$

is nonzero for $0 \leqslant i \leqslant n-1$, where $S_i = \mathrm{Syz}_i(M/\mathfrak{m}M)$.

Additionally, we give a direct proof of the above theorem. While the proof we give is similar to the proofs of [8, Theorem 3.2] and [11, Theorem 1.2], there are some illuminating differences. We will discuss this in detail later.

In Section 4.2, we would like to address the following questions: Consider a local ring $(A, \mathfrak{m})$ of dimension n where the validity of the Canonical Element Conjecture is unknown. Do there exist A-modules M such that the map

$$\mathrm{Ext}^n_A(M/\mathfrak{m}M, \mathrm{Syz}_n(M/\mathfrak{m}M)) \to \mathrm{H}^n_{\mathfrak{m}}(M, \mathrm{Syz}_n(M/\mathfrak{m}M))$$

is nonzero? Moreover, can we find A-modules M such that the map

$$\mathrm{Ext}^i_A(M/\mathfrak{m}M, \mathrm{Syz}_i(M/\mathfrak{m}M)) \to \mathrm{H}^i_{\mathfrak{m}}(M, \mathrm{Syz}_i(M/\mathfrak{m}M))$$

is nonzero even when $i > n$? We answer these questions with the following theorem:

Theorem 4.4. *Let $(A, \mathfrak{m})$ be a local ring which is not regular and let $(P_\bullet, \rho_\bullet)$ be a minimal free resolution of some finitely generated A-module Q of infinite projective dimension. If $M = \mathrm{Coker}(\rho_i^*)$ for some $i > 2$, then the canonical map*

$$\mathrm{Ext}^i_A(M/\mathfrak{m}M, S_i) \to \mathrm{H}^i_{\mathfrak{m}}(M, S_i)$$

is nonzero, where $S_i = \mathrm{Syz}_i(M/\mathfrak{m}M)$.

As a corollary, we show that in the case where A is a Gorenstein ring that is not regular, there are infinitely many isomorphism classes of A-modules M such that the map

$$\mathrm{Ext}^i_A(M/\mathfrak{m}M, \mathrm{Syz}_i(M/\mathfrak{m}M)) \to \mathrm{H}^i_{\mathfrak{m}}(M, \mathrm{Syz}_i(M/\mathfrak{m}M))$$

is nonzero for all i.

Now consider [6, Theorem 1.1]:

Theorem 1.1 (Dutta [6, Theorem 1.1]). *Let A be a ring of dimension n and depth $d < n$. Let*

$$L_\bullet : \qquad \cdots \longrightarrow L_i \xrightarrow{\lambda_i} L_{i-1} \longrightarrow \cdots \longrightarrow L_1 \xrightarrow{\lambda_1} L_0 \longrightarrow 0$$

be a minimal complex of finitely generated free A-modules such that:

(1) $H_0(L_\bullet) \neq 0$.

(2) $\ell(H_i(L_\bullet)) < \infty$ *for all i.*

(3) $H_i(L_\bullet) = 0$ *for $i \geqslant n - d$.*

Then $\operatorname{Coker}(\lambda_i)$ *cannot have a free summand if $i > 1$ and $i \neq n$. Moreover* $\operatorname{Coker}(\lambda_n)$ *cannot have a free summand if and only if the Canonical Element Conjecture holds.*

Given an arbitrary complex satisfying conditions 1, 2, and 3 above, it is extremely difficult to determine whether $\operatorname{Coker}(\lambda_n)$ has a free summand. In the following theorem we use a similar technique as used in the proof of Theorem 4.4, to prove the existence of a large class of such complexes such that $\operatorname{Coker}(\lambda_n)$ cannot have a free summand.

Theorem 4.6. *Let $(A, \mathfrak{m})$ be a local ring of dimension n and depth d such that $n - d > 1$. Then there exists infinitely many isomorphism classes of complexes of finitely generated free modules*

$$L_\bullet : \qquad \cdots \longrightarrow L_i \xrightarrow{\lambda_i} L_{i-1} \longrightarrow \cdots \longrightarrow L_1 \xrightarrow{\lambda_1} L_0 \longrightarrow 0$$

such that:

(1) $H_0(L_\bullet) \neq 0$.

(2) $\ell(H_i(L_\bullet)) < \infty$.

(3) $H_i(L_\bullet) = 0$ *for $i \geqslant n - d$.*

(4) $\operatorname{Coker}(\lambda_n)$ *does not have a free summand.*

In light of Theorem 1.1, the Canonical Element Conjecture is equivalent to proving that every complex that satisfies conditions (1), (2), and (3), also satisfies condition (4) above. While Theorem 4.6 shows that there are infinitely many isomorphism classes of complexes satisfying these conditions, we would like to find more. Currently, we are working to expand the class of complexes for which conditions (1), (2), (3), and (4), are known to hold.

2 Background

Throughout this work, $(A, \mathfrak{m}, k)$ will denote a local ring, where $\mathfrak{m}$ is the maximal ideal and $k = A/\mathfrak{m}$. Note that we will always insist that local rings are Noetherian.

2.1 Preliminary Material

We will start by giving some basic definitions and results. Each of these concepts is essential to the work that proceeds them. More information can be found in [27], [3], and [4].

2.1.1 Dimension Theory

Definition. A chain of A-modules

$$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_n = (0)$$

is a **Jordan-Hölder chain**, also known as a **composition series**, if for each i, $M_i/M_{i+1} \simeq A/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ in A.

It is not difficult to show that each composition series for M has the same length.

Definition. The **length** of an A-module, denoted by $\ell_A(M)$, is the length of a composition series for M. That is, if

$$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_n = (0)$$

is a composition series, then $\ell_A(M) = n$.

Definition. If A is a ring, the **Krull dimension** of A, denoted by $\dim(A)$, is defined as

$$\dim(A) = \sup\left\{ d : \begin{array}{l} \text{there exists } \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_d \\ \text{where each } \mathfrak{p}_i \text{ is a prime ideal of } A \end{array} \right\}.$$

Often the Krull dimension of a ring is simply referred to as the **dimension** of a ring. If M is an A-module, then

$$\dim(M) := \dim(A/\operatorname{Ann}(M)).$$

An idea closely related to the dimension of a ring is the notion of a *system of parameters*:

Definition. Let A be a local ring and M be a finitely generated A-module of dimension n. Then any sequence $x_1, \ldots, x_n \in \mathfrak{m}$ such that

$$\ell(M/(x_1, \ldots, x_n)M) < \infty$$

is called a **system of parameters** of M.

The next theorem which is part of the *Dimension Theorem*, see [27, Theorem III.B.1], makes this connection explicit:

Theorem 2.1. *Let A be a local ring and M a finitely generated A-module. If*

$$s(M) := \inf \left\{ d : \begin{array}{l} \text{there exists } x_1, \ldots, x_d \in \mathfrak{m} \text{ such} \\ \text{that } \ell(M/(x_1, \ldots, x_d)M) < \infty \end{array} \right\},$$

then

$$\dim(M) = s(M).$$

2.1.2 Homological Methods

Definition. Let A be a ring, by a **complex**, we mean a sequence of A-modules and A-module homomorphisms

$$\cdots \longrightarrow X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \longrightarrow \cdots$$

such that $d_n \circ d_{n-1} = 0$ for all $n \in \mathbb{Z}$. We denote a complex by $X_\bullet$.

Definition. If $X_\bullet$ is a complex of A-modules, then the nth **homology** of $X_\bullet$ is

$$H_n(X_\bullet) := \frac{\operatorname{Ker}(d_n)}{\operatorname{Im}(d_{n+1})}.$$

Definition. Let $X_\bullet$ and $Y_\bullet$ be two complexes over a ring A. A **map of complexes**

$$f_\bullet : X_\bullet \to Y_\bullet$$

is a collection of A-module homomorphisms such that the diagram below commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & X_{n+1} \arrow[r, "d^X_{n+1}"] \arrow[d, "f_{n+1}"'] & X_n \arrow[r, "d^X_n"] \arrow[d, "f_n"'] & X_{n-1} \arrow[r] \arrow[d, "f_{n-1}"'] & \cdots \\
\cdots \arrow[r] & Y_{n+1} \arrow[r, "d^Y_{n+1}"'] & Y_n \arrow[r, "d^Y_n"'] & Y_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

One should note that given a map of complexes, $f_\bullet : X_\bullet \to Y_\bullet$, we obtain a collection of homomorphisms:

$$\begin{aligned} f_i &: \operatorname{Ker}(d_i^X) \to \operatorname{Ker}(d_i^Y). \\ f_i &: \operatorname{Im}(d_{i+1}^X) \to \operatorname{Im}(d_{i+1}^Y). \\ H_n(f_\bullet) &: H_n(X_\bullet) \to H_n(Y_\bullet). \end{aligned}$$

From a given map of complexes, we can form a new complex, one that is crucial to the later theorems of this thesis:

Definition. Given a map of complexes $f_\bullet : X_\bullet \to Y_\bullet$, the **mapping cone** of $f_\bullet$ is the following complex:

$$\cdots \longrightarrow X_i \oplus Y_{i+1} \longrightarrow X_{i-1} \oplus Y_i \longrightarrow X_{i-2} \oplus Y_{i-1} \longrightarrow \cdots$$

where the degree i part is $X_{i-1} \oplus Y_i$ and the differentials are defined as follows:

$$\begin{aligned} d_i : X_{i-1} \oplus Y_i &\to X_{i-2} \oplus Y_{i-1} \\ (x, y) &\mapsto (-d_{i-1}^X(x), d_i^Y(y) - f_{i-1}(x)) \end{aligned}$$

Definition. Given a complex $X_\bullet$, $X_\bullet(j)$ is used to denote a **shift**, where $X_i(j) := X_{i+j}$.

Proposition 2.2. *Given a map of complexes $f_\bullet : X_\bullet \to Y_\bullet$, let $C_\bullet$ be the mapping cone of $f_\bullet$. Then there is a short exact sequence of complexes:*

$$\begin{tikzcd} 0 \arrow[r] & Y_\bullet \arrow[r] & C_\bullet \arrow[r] & X_\bullet(-1) \arrow[r] & 0 \\ & y \arrow[r, mapsto] & (0, y) & & \\ & & (x, y) \arrow[r, mapsto] & x & \end{tikzcd}$$

The above short exact sequence of complexes induces a long exact sequence:

$$\cdots \longrightarrow H_i(X_\bullet) \xrightarrow{H_i(f_\bullet)} H_i(Y_\bullet) \longrightarrow H_i(C_\bullet) \longrightarrow \cdots$$

For a proof, see [3, §2.6].

Definition. Two maps of A-complexes

$$\begin{aligned} f_\bullet &: X_\bullet \to Y_\bullet, \\ g_\bullet &: X_\bullet \to Y_\bullet, \end{aligned}$$

are called **homotopic** if there exist A-module maps $h_n : X_n \to Y_{n+1}$ such that in the diagram below

$$\begin{tikzcd}
\cdots \arrow[r] & X_{n+1} \arrow[r, "d^X_{n+1}"] \arrow[d, shift right, "f_{n+1}"'] \arrow[d, shift left, "g_{n+1}"] & X_n \arrow[r, "d^X_n"] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] \arrow[dl, "h_n"'] & X_{n-1} \arrow[r] \arrow[d, shift right, "f_{n-1}"'] \arrow[d, shift left, "g_{n-1}"] \arrow[dl, "h_{n-1}"'] & \cdots \\
\cdots \arrow[r] & Y_{n+1} \arrow[r, "d^Y_{n+1}"'] & Y_n \arrow[r, "d^Y_n"'] & Y_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

we have

$$d^Y_{n+1} \circ h_n + h_{n-1} \circ d^X_n = f_n - g_n$$

for all $n \in \mathbb{Z}$. We denote this by $f_\bullet \sim g_\bullet$.

Definition. An A-module P is **projective** if any of the following equivalent conditions are met:

(1) Given any right exact sequence $M \to N \to 0$ of A modules and a homomorphism $\varphi : P \to N$, there exits $\widetilde{\varphi} : P \to M$ such that the diagram below commutes:

$$\begin{tikzcd}
 & P \arrow[dl, dashed, "\widetilde{\varphi}"'] \arrow[d, "\varphi"] & \\
M \arrow[r, "\pi"'] & N \arrow[r] & 0
\end{tikzcd}$$

(2) $\mathrm{Hom}_A(P, -)$ is an exact functor.

(3) Every short exact sequence $0 \to M' \to M \to P \to 0$ is split exact.

(4) There is a free module F such that $F \simeq P \oplus Q$ for some A-module Q.

From [27, Proposition IV.C.20] we obtain the following:

Proposition 2.3. *If A is a local ring and M is a finitely generated A-module, then M is projective if and only if M is free.*

Definition. If M is an A-module, a **projective resolution** of M is a complex of projective modules $P_\bullet$ and a map $\pi : P_0 \to M$ such that

$$\cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\pi} M \longrightarrow 0$$

is exact.

Of particular importance are the following resolutions:

Definition. Let $(A, \mathfrak{m}, k)$ be a local ring and M be a finitely generated A-module. A complex $F_\bullet$ is a **minimal free resolution** of M if:

(1) Each F_i has finite rank.

(2) $\pi : F_0 \twoheadrightarrow M$ and $\mathrm{Ker}(\pi) \subset \mathfrak{m}F_0$.

(3) $\mathrm{Im}(d_i) = \mathrm{Ker}(d_{i-1}) \subset \mathfrak{m}F_{i-1}$.

Let $(A, \mathfrak{m}, k)$ be a local ring and M be a finitely generated A-module. In practice, one constructs a minimal free resolution of M as follows: Set

$$\begin{aligned} \beta_0 &:= \operatorname{rank}_k(M/\mathfrak{m}M), \\ S_1 &:= \operatorname{Ker}(A^{\beta_0} \to M), \end{aligned}$$

and inductively define

$$\begin{aligned} \beta_i &:= \operatorname{rank}_k(S_i/\mathfrak{m}S_i), \\ S_{i+1} &:= \operatorname{Ker}(A^{\beta_i} \to S_i). \end{aligned}$$

Now we may inductively write the exact sequences:

$$\begin{aligned} &0 \to S_1 \to A^{\beta_0} \to M \to 0, \\ &0 \to S_{i+1} \to A^{\beta_i} \to S_i \to 0. \end{aligned}$$

The integers β_i are called the ith **Betti numbers** of M and the A-modules S_i are referred to as the ith **syzygies** of M. The rather mysterious word *syzygy* means *yoke*. After putting the above exact sequences together, we can see why *syzygy* is a good term to use for the S_i's, as each S_i is connecting two free modules via A-module homomorphisms:

$$\begin{tikzcd}
0 \arrow[dr] & & 0 & & & & \\
& S_2 \arrow[ur] \arrow[dr] & & & & & \\
\cdots \arrow[ur] \arrow[rr, dashed, "d_2"] & & A^{\beta_1} \arrow[rr, dashed, "d_1"] \arrow[dr] & & A^{\beta_0} \arrow[r, "\pi"] & M \arrow[r] & 0 \\
& & & S_1 \arrow[ur] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

The d_i's above are formed by taking the composition $A^{\beta_i} \to S_i \to A^{\beta_{i-1}}$, while π is the canonical surjection. Hence we obtain a free resolution of M, that is a long exact sequence of free modules ending at M:

$$\cdots \longrightarrow A^{\beta_3} \xrightarrow{d_3} A^{\beta_2} \xrightarrow{d_2} A^{\beta_1} \xrightarrow{d_1} A^{\beta_0} \xrightarrow{\pi} M \longrightarrow 0$$

Note that the conditions that A is local and Noetherian, are critical for this construction.

The following proposition is of crucial important for the work in this thesis, for a proof see [3, §3.1]:

Proposition 2.4. *Let $f : M \to N$ be a homomorphism of A-modules. Suppose* *we have a complex $P_\bullet$, where each P_i is projective and $H_0(P_\bullet) = M$:*

$$\cdots \to P_i \to P_{i-1} \to \cdots \to P_1 \to P_0 \to 0$$

Additionally, suppose that we have an exact complex:

$$\cdots \to Q_i \to Q_{i-1} \to \cdots \to Q_1 \to Q_0 \to N \to 0$$

Then there exists a map of complexes $f_\bullet : P_\bullet \to Q_\bullet$ lifting f. Moreover, if $g_\bullet$ is another lift of the map f, then $f_\bullet \sim g_\bullet$.

Ext Modules

Consider **any** projective resolution of an A-module M:

$$\cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\pi} M \longrightarrow 0$$

Apply the functor $\mathrm{Hom}_A(-, N)$ and remove the term $\mathrm{Hom}_A(M, N)$ to get the complex $\mathrm{Hom}_A(P_\bullet, N)$:

$$0 \xrightarrow{d_0^*} \mathrm{Hom}_A(P_0, N) \xrightarrow{d_1^*} \mathrm{Hom}_A(P_1, N) \xrightarrow{d_2^*} \mathrm{Hom}_A(P_2, N) \longrightarrow \cdots$$

where $d_0^* := 0$. We now define:

$$\mathrm{Ext}^i_A(M, N) := H^i(\mathrm{Hom}_A(P_\bullet, N)) = \frac{\mathrm{Ker}(d^*_{i+1})}{\mathrm{Im}(d^*_i)}$$

Proposition 2.5. *Given a ring A and an A-module N, $\boldsymbol{\mathrm{Ext}^i_A(-, N)}$ is the left derived functor of the left exact contravariant functor $\mathrm{Hom}_A(-, N)$. Explicitly, given an exact sequence of A-modules,*

$$0 \to M' \to M \to M'' \to 0$$

we obtain a long exact sequence of Ext*'s:*

$$0 \to \mathrm{Ext}^0_A(M'', N) \to \mathrm{Ext}^0_A(M, N) \to \mathrm{Ext}^0_A(M', N) \to \mathrm{Ext}^1_A(M'', N) \to \cdots$$

For a proof, see [3, §5.4].

Definition. Given a ring A and an A-module M, the **projective dimension** of M is defined to be:

$$\mathrm{pd}_A(M) := \inf\{n : \text{there exists a projective resolution of } M \text{ of length } n\}.$$

Recall that

$$0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

is a resolution of length n if it is an exact complex and each P_i is projective.

Proposition 2.6. *If A is a ring and M is an A-module, then the following are equivalent:*

1. $\mathrm{pd}_A(M) \leqslant n$.
2. $\mathrm{Ext}^i_A(M, N) = 0$ *for all A-modules N and for all $i > n$.*
3. $\mathrm{Ext}^{n+1}_A(M, N) = 0$ *for all A-modules N.*

For a proof, see [27, Section IV.C].

Definition. An A-module E is **injective** if any of the following equivalent conditions are met:

1. Given any left exact sequence $0 \to M' \to M$ of A-modules and a homomorphism $\varphi : M' \to E$, there exits $\widetilde{\varphi} : M \to E$ such that the diagram below commutes:

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r, "\iota"] \arrow[d, "\varphi"'] & M \arrow[dl, dashed, "\widetilde{\varphi}"] \\ & E & \end{tikzcd}$$

2. $\mathrm{Hom}_A(-, E)$ is an exact functor.
3. Every short exact sequence $0 \to E \to M \to M'' \to 0$ is split exact.

Definition. Given a ring A and an A-module M, the **injective dimension** of M is defined to be:

$$\mathrm{id}_A(M) := \inf\{n : \text{there exists an injective resolution of } M \text{ of length } n\}.$$

Recall that

$$0 \to M \to E^0 \to E^1 \to \cdots \to E^{n-1} \to E^n \to 0$$

is a resolution of length n if it is an exact chain complex and each E^i is injective.

Proposition 2.7. *If A is a ring and N is an A-module, then the following are equivalent:*

1. $\mathrm{id}_A(N) \leqslant n$.
2. $\mathrm{Ext}^i_A(M, N) = 0$ *for all A-modules M and for all $i > n$.*
3. $\mathrm{Ext}^{n+1}_A(M, N) = 0$ *for all A-modules M.*
4. $\mathrm{Ext}^{n+1}_A(M, N) = 0$ *for all finitely generated A-modules M.*

For a proof, see [27, Section IV.C].

Definition. Given a ring A and an A-module M, $x_1, \ldots, x_n \in A$ is called an ***M*-sequence** if the following hold:

1. We have that $(x_1, \ldots, x_n)M \neq M$.

(2) The element x_1 is a nonzerodivisor on M.

(3) For each $i > 1$,

$$\frac{M}{(x_1, \dots, x_{i-1})M} \xrightarrow{x_i} \frac{M}{(x_1, \dots, x_{i-1})M}$$

is an injective map; that is, x_i is a nonzerodivisor on $M/(x_1, \dots, x_{i-1})M$ for $1 \leqslant i \leqslant n$.

Proposition 2.8. *Let A be a Noetherian ring, M be a finitely generated A-module, and I be an ideal of A such that $IM \neq M$. The following are equivalent:*

(1) $\operatorname{Ext}^i_A(N, M) = 0$ *for all $i < n$ and for all finitely generated A-modules N such that* $\operatorname{Supp}_A(N) \subset V(I) = \{\mathfrak{p} \in \operatorname{Spec}(A) : \mathfrak{p} \supset I\}$.

(2) $\operatorname{Ext}^i_A(N, M) = 0$ *for all $i < n$ for some finitely generated A-module N where* $\operatorname{Supp}_A(N) = V(I)$.

(3) *There exist $x_1, \dots, x_n \in I$ which form an M-sequence.*

For a proof, see [4, Proposition 1.2.10].

Definition. Let A be a Noetherian ring, M a finitely generated A-module, and I an ideal of A such that $IM \neq M$. Then we define the ***I*-depth** of a module as:

$$\begin{aligned} \operatorname{depth}_I(M) &:= \inf\{i : \operatorname{Ext}^i(A/I, M) \neq 0\}, \\ &= \text{length of maximal } M\text{-sequence in } I. \end{aligned}$$

If A is a local ring with maximal ideal $\mathfrak{m}$, then $\operatorname{depth}_{\mathfrak{m}}(M)$ will be abbreviated by $\operatorname{depth}(M)$.

The next theorem connects the notion of depth and the notion of projective dimension:

Theorem 2.9 (Auslander-Buchsbaum Formula). *If A is a local ring and M is a nonzero finitely generated A-module of finite projective dimension, then*

$$\operatorname{pd}(M) + \operatorname{depth}(M) = \operatorname{depth}(A).$$

For a proof, see [4, Theorem 1.3.3]

Proposition 2.10. *If*

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is a short exact sequence of finitely generated A-modules. Set $d_i = \operatorname{depth}_I(M_i)$. Then:

(1) $d_1 \geqslant \min\{d_2, d_3 + 1\}$.

(2) $d_2 \geqslant \min\{d_1, d_3\}$.

(3) $d_3 \geqslant \min\{d_1 - 1, d_2\}$.

For a proof, see [27, Proposition IV.B.8]

The next proposition tells us the relationship between an M-sequence and a system of parameters:

Proposition 2.11. *If A is a local ring, every M-sequence is part of a system of parameters for M.*

For a proof, see [27, Proposition IV.A.7]. Thus we see that given a local ring $\operatorname{depth}(M) \leqslant \dim(M)$. At this point we can describe some important types of rings.

Definition. If $(A, \mathfrak{m})$ is local, A is a **regular local ring** if

$$\dim(A) = \boldsymbol{\mu}(\mathfrak{m}) = \{\text{the minimal number of generators of } \mathfrak{m}\}.$$

Example 2.12. *Let k be a field and V be a discrete valuation ring, then both*

$$k[\![x_1, \dots, x_n]\!] \qquad \text{and} \qquad V[\![x_1, \dots, x_n]\!]$$

are regular local rings.

Definition. A local ring is called a **complete intersection** if it is of the form

$$A/(x_1, \dots, x_r)$$

where A is a regular ring and $x_1, \dots, x_r$ an A-sequence.

Example 2.13. *Let k be a field, then*

$$\frac{k[\![x_1, x_2]\!]}{(x_1^7)}$$

is a complete intersection ring.

Definition. A local ring is called a **Gorenstein ring** if it has finite injective dimension as a module over itself.

Example 2.14. *Let k be a field, then $k[\![x^2, x^3]\!]$ is Gorenstein.*

Definition. A local ring is called **Cohen-Macaulay** if its depth is equal to its dimension.

Example 2.15. *Let k be a field, then $k[\![x^3, x^4, x^5]\!]$ is Cohen-Macaulay.*

We have the following implications for local rings:

Regular Local ⇒ Complete Intersection ⇒ Gorenstein ⇒ Cohen-Macaulay

None of these implications can be reversed, see [4, Chapter 3].

2.1.3 Local Cohomology

In the 1950's, Grothendieck developed a powerful tool called local cohomology. We will give a very brief description of local cohomology modules and their properties. More information can be found in [14].

Definition. Let A be a Noetherian ring, M an A-module and I an ideal of A. We define the ith **local cohomology** of M with respect to I by

$$\mathrm{H}^i_I(M) := \varinjlim_t \mathrm{Ext}^i_A(A/I^t, M).$$

Unfortunately this definition tells us next to nothing about what local cohomology really is. Here is a more understandable definition:

Definition. Let A be a Noetherian ring, M an A-module and I an ideal of A. We define the ith **local cohomology** of M with respect to I as the ith derived functor of the left exact functor:

$$\Gamma_I(M) = \{x \in M : I^t x = 0 \text{ for some } t > 0\}.$$

Explicitly, this means that given an exact sequence of A-modules,

$$0 \to M' \to M \to M'' \to 0$$

we obtain a long exact sequence of local cohomology modules with $\mathrm{H}^0_{\mathfrak{m}}(M) = \Gamma_I(M)$:

$$0 \to \mathrm{H}^0_{\mathfrak{m}}(M'') \to \mathrm{H}^0_{\mathfrak{m}}(M) \to \mathrm{H}^0_{\mathfrak{m}}(M') \to \mathrm{H}^1_{\mathfrak{m}}(M'') \to \cdots$$

Theorem 2.16 (Vanishing). *If $(A, \mathfrak{m})$ is local and M is a finitely generated A-module, then*

$$\inf\{i : \mathrm{H}^i_{\mathfrak{m}}(M) \neq 0\} = \mathrm{depth}(M) \leqslant \dim(M) = \sup\{i : \mathrm{H}^i_{\mathfrak{m}}(M) \neq 0\}.$$

For a proof, see [14, Theorem 3.8 and Proposition 6.4].

2.2 The Homological Conjectures

2.2.1 History and Origins

The Homological Conjectures are a set of conjectures and theorems that have their roots in the works of Serre [27], Auslander [1], Bass [2], Peskine and

Szpiro [23], and Hochster [19]. In his classic text, *Local Algebra*, Serre proves the following theorem:

Theorem 2.17 (Serre [27, Chapter V]). *If A is a regular local ring with finitely generated A-modules M and N such that $\ell(M \otimes_A N)$ is finite, then*

$$\dim(M) + \dim(N) \leqslant \dim(A).$$

Recall that the condition $\ell(M \otimes_A N) < \infty$ is equivalent to saying that $\dim(M \otimes_A N) = 0$, which is equivalent to $\operatorname{Supp}(M) \cap \operatorname{Supp}(N) = \{\mathfrak{m}\}$. We should also remark that it is difficult to relax the condition that A is regular and still obtain the same result. To see this, consider the complete intersection ring

$$A = k[\![X,Y,U,V]\!]/(XU - YV), \qquad M = A/(X,Y), \qquad N = A/(U,V).$$

Now $\dim(M) = \dim(N) = 2$ but $\dim(A) = 3$, and so the the theorem above does not hold.

In [23], Peskine and Szpiro generalized this theorem. Recall the Auslander-Buchsbaum Formula, Theorem 2.9. By Serre's Theorem, if A is a regular local ring and $\ell(M \otimes_A N)$ is finite, then

$$\operatorname{pd}(M) + \operatorname{depth}(M) \geqslant \dim(M) + \dim(N).$$

Thus we see that:

$$\operatorname{pd}(M) \geqslant \dim(N)$$

Thus we are led to the Intersection Theorem, first stated in [23] by Peskine and Szpiro:

Theorem 2.18 (The Intersection Theorem). *Let A be a local ring and let M and N be finitely generated A-modules where M has finite projective dimension and $M \otimes_A N$ has finite length, then* $\dim(N) \leqslant \operatorname{pd}(M)$.

In [23], Peskine and Szpiro proved the Intersection Theorem in the following cases:

1. The characteristic of A is a prime number p.
2. A is essentially of finite type over a field of characteristic 0.
3. A is ind-étale over a ring which is essentially of finite type over a field of characteristic 0.

Recall that A is ind-étale when:

1. For each α, B_α is local and essentially finite type over a field.
2. $B_\alpha \to B'_\alpha$ is the localization of an étale map.

(3) $A = \varinjlim B_\alpha$.

Moreover, Peskine and Szpiro showed that when the conclusion of the Intersection Theorem holds, the following conjectures of Auslander [1, Section 4] and Bass [2, Section 3] are answered in the affirmative:

Conjecture (Auslander). *Let A be a local ring. If M is a nonzero finitely generated A-module of finite projective dimension, then every M-sequence is an A-sequence.*

Conjecture (Bass). *If A is local and M is a nonzero finitely generated A-module such that $\mathrm{id}(M) < \infty$, then A is Cohen-Macaulay.*

Now if we consider a minimal free resolution of a finitely generated module over a ring for which the Intersection Theorem holds, and a finitely generated A-module M such that $\ell(M \otimes_A A/\mathfrak{p}) < \infty$, we may write

$$0 \to A^{t_n} \to \cdots \to A^{t_0} \to M \to 0$$

and then apply $- \otimes_A A/\mathfrak{p}$ to obtain a finite complex of finitely generated free $\overline{A}$ modules of length $n = \mathrm{pd}(M)$

$$F_\bullet : \qquad 0 \to \overline{A}^{t_n} \to \cdots \to \overline{A}^{t_0} \to 0.$$

Since $\ell(M \otimes_A A/\mathfrak{p}) < \infty$, we have

$$\ell(H_i(F_\bullet)) = \ell(\mathrm{Tor}_i^A(M, A/\mathfrak{p})) < \infty,$$

then by the Intersection Theorem $\dim(A/\mathfrak{p}) \leqslant n$. Hence we are led to the New Intersection Theorem, first stated by Peskine and Szpiro in [24]:

Theorem 2.19 (New Intersection Theorem). *Let $(A, \mathfrak{m})$ be a local ring and*

$$F_\bullet : \qquad 0 \to F_n \to \cdots \to F_0 \to 0$$

be a complex of finitely generated free modules with $H_0(F_\bullet) \neq 0$. If $\ell(H_i(F_\bullet))$ is finite for $i \geqslant 0$, then $\dim(A) \leqslant n$.

Around 1975, Peskine and Szpiro [24], and Roberts [25], independently proved the New Intersection Theorem for all local rings of positive characteristic. In [20], Hochster proved the conjecture for all rings containing a field. Later in [26], Roberts used Intersection Theory *a la* Fulton to prove the theorem in the mixed characteristic case. Since the New Intersection Theorem implies the Intersection Theorem, this settles the validity of the Intersection Theorem for all local rings.

In [12], Evans and Griffith implicitly proved what is known as the Improved New Intersection Conjecture for all local rings containing a field. It was stated explicitly by Hochster in [21]:

Conjecture (Improved New Intersection Conjecture). *Let* $(A, \mathfrak{m})$ *be a local ring and*

$$F_\bullet : \qquad 0 \to F_n \to \cdots \to F_0 \to 0$$

be a complex of finitely generated free modules. If $\ell(H_i(F_\bullet)) < \infty$ *for* $i > 0$ *and* $H_0(F_\bullet)$ *has a nonzero minimal generator killed by a power of* $\mathfrak{m}$*, then* $\dim(A) \leqslant n$*.*

It is clear that the Improved New Intersection Conjecture (INIC) implies the New Intersection Conjecture (NIC). Hence from all this we see that:

$$\begin{tikzcd}
 & & & \text{(Auslander)} \\
\text{(INIC)} \arrow[r, Rightarrow] & \text{(NIT)} \arrow[r, Rightarrow] & \text{(IT)} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & \\
 & & & \text{(Bass)}
\end{tikzcd}$$

Where (IT) stands for the Intersection Theorem, (Bass) stands for Bass' Conjecture, and (Auslander) stands for Auslander's Conjecture.

2.2.2 The Equational Side

If A is a normal domain of characteristic zero containing the rationals and B is a module finite extension of A, $\iota : A \hookrightarrow B$, then ι splits as an A-module via the trace map. In [19], Hochster proves a similar situation for regular local rings of positive characteristic and conjectures the following:

Conjecture (Direct Summand Conjecture). *If* $A \subset B$ *are rings, where* A *is a regular local ring and* B *is a module finite* A*-algebra, then* A *is a direct summand of* B*. In other words* $A \hookrightarrow B$ *splits as an* A*-module map.*

In [19], Hochster points out that it is difficult to relax the condition that A is a regular local ring in the conjecture above. For example, consider $A = k[u^2, v^2, u^3 + v^3]$, where k is an algebraically closed field of characteristic 2. Constructing a k-homomorphism

$$\varphi : k[x, y, z] \twoheadrightarrow A,$$

via

$$x \mapsto u^2, \qquad y \mapsto v^2, \qquad z \mapsto u^3 + v^3,$$

we have $\operatorname{Ker}(\varphi) = (x^3 + y^3 - z^2)$, so we have that A is a complete intersection ring and it is seen easily that A is integrally closed. However, in A

$$u^3 + v^3 \notin (u^2, v^2)A$$

but in $k[u,v]$, an integral extension of A,

$$(u^3+v^3) \in (u^2,v^2)k[u,v].$$

Thus (u^2,v^2) is not a contracted ideal and hence A is not a direct summand of $k[u,v]$.

In [19], Hochster proved the Direct Summand Conjecture in the equicharacteristic case. Moreover, Hochster showed that the Direct Summand Conjecture is true if and only if the equation

$$(X_1 \cdots X_n)^{t-1} - \sum_{i=1}^{n} Y_i X_i^t = 0$$

has no solutions $(x_1,\dots,x_n,y_1,\dots,y_n)$ in any local ring B such that $x_1,\dots,x_n$ is a system of parameters for B, where $t \geqslant 1$. In [20], Hochster states this formally as the Monomial Conjecture:

Conjecture (Monomial Conjecture). *If B is a local ring with a system of parameters $x_1,\dots,x_n$ then for all $t \geqslant 1$*

$$(x_1 \cdots x_n)^{t-1} \notin (x_1^t,\dots,x_n^t).$$

In [21], Hochster proved that the Direct Summand Conjecture, and hence the Monomial Conjecture, implies the Improved New Intersection Conjecture. These statements demonstrate why we refer to this section as the *equational side* of the Homological Conjectures.

2.2.3 A Further Abstraction

In the 1970's, Hochster formulated the following conjecture:

Conjecture (The Canonical Element Conjecture). *Let $(A,\mathfrak{m},k)$ be a local ring, $x_1,\dots,x_n$ a system of parameters, and $P_\bullet$ a projective resolution of k. If $\varphi_\bullet$ denotes a lift of the canonical surjection $\pi : A/\mathbf{x} \to k$, then $\varphi_n : A \to P_n$ is nonzero.*

$$\begin{tikzcd}
0 \arrow[r] & K_n(\mathbf{x}) \arrow[r] \arrow[d, "\varphi_n"] & \cdots \arrow[r] & K_0(\mathbf{x}) \arrow[r] \arrow[d, "\varphi_0"] & A/\mathbf{x} \arrow[r] \arrow[d, two heads, "\pi"] & 0 \\
\cdots \arrow[r] & P_n \arrow[r] & \cdots \arrow[r] & P_0 \arrow[r] & k \arrow[r] & 0
\end{tikzcd}$$

Note that this result is trivial if A is regular. Henceforth we will assume that A is not regular. Since given any two projective resolutions $P_\bullet$ and $Q_\bullet$ of k there are maps $P_\bullet \to Q_\bullet \to P_\bullet$ which lift to the identity, it suffices to consider a minimal resolution of k. Moreover, if $\mathbf{x}$ is a system of parameters, then so is

$\mathbf{x}^t$. Consider the following diagram, where $F_\bullet$ is a minimal free resolution of k:

$$\begin{tikzcd}
0 \arrow[r] & K_n(\mathbf{x}) \arrow[r] \arrow[d, "\varphi_n"] & \cdots \arrow[r] & K_0(\mathbf{x}) \arrow[r] \arrow[d, "\varphi_0"] & A/\mathbf{x} \arrow[r] \arrow[d, two heads, "\pi"] & 0 \\
\cdots \arrow[r] & F_n \arrow[r] & \cdots \arrow[r] & F_0 \arrow[r] & k \arrow[r] & 0
\end{tikzcd}$$

Examining the above diagram near degree n and setting $S = \mathrm{Syz}_n(k)$ we have:

$$\begin{tikzcd}
0 \arrow[r] & K_n(\mathbf{x}^t) \arrow[rr] \arrow[d, "\varphi_n"] & & K_{n-1}(\mathbf{x}^t) \arrow[r] \arrow[d, "\varphi_{n-1}"] & \cdots \\
\cdots \arrow[r] & F_n \arrow[rr, "d_n"] \arrow[dr, two heads] & & F_{n-1} \arrow[r] & \cdots \\
& & S \arrow[ur, hook] & &
\end{tikzcd}$$

Applying $\mathrm{Hom}_A(-, S)$ to everything we obtain:

$$\begin{tikzcd}
0 & \mathrm{Hom}_A(K_n(\mathbf{x}^t), S) \arrow[l] & & \mathrm{Hom}_A(K_{n-1}(\mathbf{x}^t), S) \arrow[ll] & \cdots \arrow[l] \\
\cdots & \mathrm{Hom}_A(F_n, S) \arrow[l] \arrow[u, "\varphi_n^*"] & & \mathrm{Hom}_A(F_{n-1}, S) \arrow[ll, "d_n^*"'] \arrow[u, "\varphi_{n-1}^*"] \arrow[dl] & \cdots \arrow[l] \\
& & \mathrm{Hom}_A(S, S) \arrow[ul, hook] & &
\end{tikzcd}$$

Look at:

$$\mathrm{Ext}_A^n(k, S) = \frac{\mathrm{Ker}(d_{n+1}^*)}{\mathrm{Im}(d_n^*)}$$

Could $\mathbb{1}_S \in \mathrm{Hom}_A(S, S)$ be in $\mathrm{Im}(d_n^*)$? If so then we have a retract:

$$\begin{tikzcd}
& F_{n-1} \arrow[dr] & \\
S \arrow[ur] \arrow[rr, bend right, "\mathbb{1}_S"'] & & S
\end{tikzcd}$$

Thus we may truncate our minimal resolution of k at the nth spot, showing that $\mathrm{Tor}_{n+1}(k, k) = 0$ and thus our ring is regular, a contradiction. Hence we see that:

$$\begin{aligned}
\mathrm{Ext}_A^n(k, S) &\to \mathrm{Ext}_A^n(A/\mathbf{x}^t, S) \\
\overline{\mathbb{1}}_S &\mapsto \text{a nonzero element}
\end{aligned}$$

In [21], Hochster shows that this line of reasoning leads to what turns out to be an equivalent way of stating the Canonical Element Conjecture:

Conjecture (Equivalent Verison I). *If $(A, \mathfrak{m}, k)$ is a local ring and $S =$*

$\mathrm{Syz}_n(k)$, *then the image of* $\mathbb{1}_S$ *in* $\mathrm{Ext}^n_A(k, S)$ *is nonzero in* $\mathrm{Ext}^n_A(A/\mathbf{x}^t, S)$. *Thus*

$$\mathrm{Ext}^n_A(k, S) \to \varinjlim_t \mathrm{Ext}^n_A(A/\mathfrak{m}^t, S) = \mathrm{H}^n_{\mathfrak{m}}(S)$$

is nonzero. The image of $\mathbb{1}_S$ *in* $\mathrm{H}^n_{\mathfrak{m}}(S)$ *is usually denoted* η_A *and is called the canonical element.*

In [5], Dutta presents yet another way of thinking about the Canonical Element Conjecture:

Conjecture (Equivalent Version II). *Let* $(A, \mathfrak{m})$ *be a local ring,* $F_\bullet$ *be a minimal free resolution of* $A/\mathfrak{m}$, *and* $G_\bullet$ *be a minimal free resolution of* $A/\mathfrak{m}^t$. *If* $\varphi_\bullet$ *denotes a lift of the canonical surjection* $\pi : A/\mathfrak{m}^t \to A/\mathfrak{m}$, *then* $\varphi_n : G_n \to F_n$ *is nonzero.*

$$\begin{tikzcd}
\cdots \arrow[r] & G_n \arrow[r] \arrow[d, "\varphi_n"] & \cdots \arrow[r] & G_0 \arrow[r] \arrow[d, "\varphi_0"] & A/\mathfrak{m}^t \arrow[r] \arrow[d, two heads, "\pi"] & 0 \\
\cdots \arrow[r] & F_n \arrow[r] & \cdots \arrow[r] & F_0 \arrow[r] & A/\mathfrak{m} \arrow[r] & 0
\end{tikzcd}$$

At this point we will briefly sketch the proof that if a local ring $(A, \mathfrak{m})$ is Cohen-Macaulay, then the Canonical Element Conjecture holds for A. Suppose that the Canonical Element Conjecture did not hold for A. Since A is a Cohen-Macaulay ring, we obtain the following commutative diagram with exact rows where $L_\bullet$ is a minimal free resolution of $\mathrm{Ext}^{n-1}_A(A/\mathfrak{m}^t, A)$:

$$\begin{tikzcd}
L_n \arrow[r, "\lambda_n"] \arrow[d] & L_{n-1} \arrow[r, "\lambda_{n-1}"] \arrow[d, "\psi_{n-1}"] & L_{n-2} \arrow[r] \arrow[d, "\psi_{n-2}"] & \cdots \arrow[r] & L_0 \arrow[r] \arrow[d, "\psi_0"] & \mathrm{Ext}^{n-1}_A(A/\mathfrak{m}^t, A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] \arrow[d] & G_0^* \arrow[r] \arrow[d] & G_1^* \arrow[r] \arrow[d] & \cdots \arrow[r] & G_{n-1}^* \arrow[r] \arrow[d] & G_{n-1}^*/\operatorname{Im}(\tau_{n-1}^*) \arrow[r] \arrow[d] & 0 \\
L_{n-1} \arrow[r] & L_{n-2} \oplus G_0^* \arrow[r] & L_{n-3} \oplus G_1^* \arrow[r] & \cdots \arrow[r] & G_{n-1}^* \arrow[r] & \operatorname{Im}(\tau_n^*) \arrow[r] & 0
\end{tikzcd}$$

Then with notation as in the conjecture above, one could take $\varphi_i = 0$ for $i \geqslant n$ and show that the following composition is zero:

$$F_{n-1}^*/\operatorname{Im}(\sigma_{n-1}^*) \to \mathrm{Ext}^{n-1}_A(M/\mathfrak{m}^t M, A) \to G_{n-1}^*/\operatorname{Im}(\tau_{n-1}^*) \to \operatorname{Im}(\tau_n^*)$$

Moreover, since the above composition is zero, we obtain homotopy maps:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & F_0^* \arrow[r, "\sigma_0^*"] \arrow[d, equal] \arrow[ddl, dashed, "h"'] & F_1^* \arrow[r] \arrow[d] \arrow[ddl, dashed, "h'"'] & \cdots \arrow[r] & F_{n-1}^* \arrow[r] \arrow[d] & F_{n-1}^*/\operatorname{Im}(\sigma_{n-1}^*) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] \arrow[d] & G_0^* \arrow[r] \arrow[d, "(0,\mathbb{1}_{G_0^*})"'] & G_1^* \arrow[r] \arrow[d] & \cdots \arrow[r] & G_{n-1}^* \arrow[r] \arrow[d] & G_{n-1}^*/\operatorname{Im}(\tau_{n-1}^*) \arrow[r] \arrow[d] & 0 \\
L_{n-1} \arrow[r] & L_{n-2} \oplus G_0^* \arrow[r] & L_{n-3} \oplus G_1^* \arrow[r] & \cdots \arrow[r] & G_{n-1}^* \arrow[r] & \operatorname{Im}(\tau_n^*) \arrow[r] & 0
\end{tikzcd}$$

This diagram can be used to show that $\operatorname{Im}(\psi_{n-1}) = G_0^*$. However, by our first diagram above, this will show that $\operatorname{Coker}(\lambda_n)$ has a free summand. This cannot be as $L_\bullet$ is a minimal free resolution. The interested reader should compare the above sketch to the proof given in [21, Theorem 2.7].

Hochster was interested in the Canonical Element Conjecture because he could show that it implied all of the above mentioned conjectures, see [21]. In [5], Dutta showed that the Improved New Intersection Conjecture implied the Canonical Element Conjecture. Thus, the relationship between the conjectures above is as follows

$$\begin{tikzcd}
\text{(MC)} \arrow[r, Leftrightarrow] \arrow[d, Leftrightarrow] & \text{(DSC)} \arrow[d, Leftrightarrow] \\
\text{(CEC)} \arrow[r, Leftrightarrow] & \text{(INIC)}
\end{tikzcd}$$

Here (MC) stands for the Monomial Conjecture, (DSC) stands for Direct Summand Conjecture, (CEC) stands for the Canonical Element Conjecture, and (INIC) stands for the Improved New Intersection Conjecture. In [21, Proposition 3.18, Remark 3.19], it is shown that the Canonical Element Conjecture can be reduced to the case of a complete local normal domain. These conjectures were proved in the equicharacteristic case by Evans and Griffith in [12] and by Hochster in [21]. In the mixed characteristic case, special cases are shown in [13, 22, 5, 6, 7, 8, 9, 10, 11]. Recently, in [16] Heitmann proved the conjectures in the case where the ring is of dimension three.

In [8] and [11], Dutta takes a slightly different view. Given a local ring $(A, \mathfrak{m})$ of dimension n, he chooses to study the canonical maps:

$$\operatorname{Ext}^i_A(A/\mathfrak{m}, \operatorname{Syz}_i(A/\mathfrak{m}) \to \operatorname{H}^i_{\mathfrak{m}}(\operatorname{Syz}(A/\mathfrak{m}))$$

In [8] and [11], Dutta gives two different proofs of the following theorem:

Theorem 2.20 (Dutta). *Let $(A, \mathfrak{m})$ be a local ring. The canonical map*

$$\operatorname{Ext}^i_A(A/\mathfrak{m}, S_i) \to \operatorname{H}^i_{\mathfrak{m}}(S_i)$$

is nonzero for $0 \leqslant i \leqslant n-1$, where $S_i = \operatorname{Syz}_i(A/\mathfrak{m})$.

It was this way of thinking that motivated much of the work of this thesis.

3 Generalized Local Cohomology

3.1 Basic Results

Generalize the idea of local cohomology modules as follows:

Definition. If $(A, \mathfrak{m})$ is a local ring with finitely generated A-modules M and S, the ith ***M*-local cohomology** of S with respect to $\mathfrak{m}$ is defined as:

$$\mathrm{H}^i_{\mathfrak{m}}(M, S) := \varinjlim_t \mathrm{Ext}^i_A(M/\mathfrak{m}^t M, S).$$

While these modules were first studied by Grothendieck in [15, Exposé VI], they were not explicitly called *generalized local cohomology* modules. In his thesis [17], Herzog seems to be the first to view these modules as a generalization of local cohomology modules. These modules have attracted the interest of others as well.

First we should note that

$$\begin{aligned}\mathrm{H}^0_{\mathfrak{m}}(M, N) &= \varinjlim \mathrm{Hom}_A(M/\mathfrak{m}^t M, N)\\ &\simeq \varinjlim \mathrm{Hom}_A(A/\mathfrak{m}^t, \mathrm{Hom}_A(M, N))\\ &= \mathrm{H}^0_{\mathfrak{m}}(\mathrm{Hom}_A(M, N)).\end{aligned}$$

Since $\mathrm{Hom}(M, -)$ and $\mathrm{H}^0_{\mathfrak{m}}(-)$ are left exact, we have that $\mathrm{H}^0_{\mathfrak{m}}(M, -)$ is a left exact functor. To compute $\mathrm{H}^i_{\mathfrak{m}}(M, N)$, one could consider an injective resolution of N, $0 \to N \to I^\bullet$, apply the functor of $\mathrm{H}^0_{\mathfrak{m}}(M, -)$, and take the cohomology. To see this write:

$$\begin{aligned}H^i(\varinjlim \mathrm{Hom}_A(M/\mathfrak{m}^t M, I^\bullet)) &= \varinjlim H^i(\mathrm{Hom}_A(M/\mathfrak{m}^t M, I^\bullet))\\ &= \varinjlim \mathrm{Ext}^i_A(M/\mathfrak{m}^t M, N)\\ &= \mathrm{H}^i_{\mathfrak{m}}(M, N).\end{aligned}$$

Along these lines we have the following proposition [28, Proposition 2.1]:

Proposition 3.1. *If M is a finitely generated A-module and N is any A-module, and $I^\bullet$ is a minimal injective resolution of N, then*

$$\mathrm{H}^i_{\mathfrak{m}}(M, N) \simeq H^i(\mathrm{H}^0_{\mathfrak{m}}(\mathrm{Hom}_A(M, I^\bullet))) \simeq H^i(\mathrm{Hom}_A(M, \mathrm{H}^0_{\mathfrak{m}}(I^\bullet))).$$

We would like to know what facts from the standard theory of local cohomology carry over to the generalized case. We are most interested in the vanishing of generalized local cohomology modules. We have some analogue of Theorem 2.16 from [18, Theorem 3.2] and [28, Theorem 2.3]:

Theorem 3.2 (Vanishing). *If M and N are finitely generated A-modules, then*

$$\inf\{i : \mathrm{H}^i_{\mathfrak{m}}(M,N) \neq 0\} = \mathrm{depth}_A(N).$$

If in addition, $\mathrm{pd}_A(M) < \infty$ *or* $\mathrm{id}_A(M) < \infty$, *then*

$$\sup\{i : \mathrm{H}^i_{\mathfrak{m}}(M,N) \neq 0\} \leqslant \dim(A).$$

The second statement above can fail in at least two ways. Let $(A, \mathfrak{m}, k)$ be a local ring with finitely generated A-modules M and N. If M does not have finite projective dimension, then let $I^\bullet$ be a minimal injective resolution of k. By Proposition 3.1

$$\mathrm{H}^i_{\mathfrak{m}}(M,k) \simeq H^i(\mathrm{Hom}_A(M, \mathrm{H}^0_{\mathfrak{m}}(I^\bullet))).$$

However, since $\mathrm{Supp}(k) = \mathfrak{m}$, we have that $\mathrm{H}^0_{\mathfrak{m}}(I^i) = I^i$ and so

$$\mathrm{H}^i_{\mathfrak{m}}(M,k) \simeq \mathrm{Ext}^i_A(M,k) \neq 0 \text{ for all } i.$$

On the other hand, if N does not have finite injective dimension, then let $I^\bullet$ be a minimal injective resolution of N. By Proposition 3.1 we have

$$\mathrm{H}^i_{\mathfrak{m}}(k,N) \simeq H^i(\mathrm{Hom}_A(k, \mathrm{H}^0_{\mathfrak{m}}(I^\bullet))).$$

But since $\mathrm{Ass}(\mathrm{Hom}_A(k, E(A/\mathfrak{p}))) = \varnothing$ except when $\mathfrak{p} = \mathfrak{m}$, we see that

$$\begin{aligned} H^i(\mathrm{Hom}_A(k, \mathrm{H}^0_{\mathfrak{m}}(I^\bullet))) &= H^i(\mathrm{Hom}_A(k, I^\bullet)) \\ &= \mathrm{Ext}^i_A(k,N), \end{aligned}$$

and $\mathrm{Ext}^i_A(k,N)$ is nonzero for infinitely many integers i.

It is clear that $\mathrm{H}^0_{\mathfrak{m}}(M,-)$ is a covariant left exact functor with an associated long exact sequence of generalized local cohomology modules. However, it is also true that $\mathrm{H}^0_{\mathfrak{m}}(-,N)$ is a contravariant left exact functor with an associated long exact sequence of generalized local cohomology modules. This is stated in [18], but here we provide a different proof.

Proposition 3.3. *Let A be a Noetherian ring, I be an ideal, and N be an A-module. Given a short exact sequence of finitely generated A-modules*

$$0 \to M' \to M \to M'' \to 0$$

we obtain the long exact sequence:

$$0 \to \mathrm{H}^0_I(M'', N) \to \mathrm{H}^0_I(M, N) \to \mathrm{H}^0_I(M', N) \to \mathrm{H}^1_I(M'', N) \to \cdots$$

Proof For two natural numbers s and t, apply $- \otimes_A A/I^{s+t}$ to the short exact sequence of finitely generated A-modules

$$0 \to M' \to M \to M'' \to 0$$

to get the exact sequence

$$0 \to K \to M'/I^{s+t}M' \to M/I^{s+t}M \to M''/I^{s+t}M'' \to 0,$$

where K is the kernel of the map from $M'/I^{s+t}M'$ to $M/I^{s+t}M$. Note that

$$K = \frac{M' \cap I^{s+t}M}{I^{s+t}M'}.$$

Break off the following exact sequences

$$\begin{aligned} &0 \to K \to M'/I^{s+t}M' \to Z \to 0, \\ &0 \to Z \to M/I^{s+t}M \to M''/I^{s+t}M'' \to 0, \end{aligned}$$

to see that:

$$Z \simeq \frac{M'/I^{s+t}M'}{(M' \cap I^{s+t}M)/I^{s+t}M'} \simeq \frac{M'}{M' \cap I^{s+t}M}$$

By the Artin-Rees Lemma we see that for some s and all t we have

$$M' \cap I^{s+t}M = I^t(M' \cap I^sM).$$

Additionally, for any given t, there exists some r such that

$$I^rM' \subset M' \cap I^{s+t}M = I^t(M' \cap I^sM) \subset I^tM'.$$

Hence the sets $I^t(M' \cap I^sM)$ and I^tM' are cofinal with respect to t. Thus we see that

$$\begin{aligned} \varinjlim_t \operatorname{Ext}^i_A(Z, N) &\simeq \varinjlim_t \operatorname{Ext}^i_A(M'/I^t(M' \cap I^sM), N) \\ &\simeq \varinjlim_t \operatorname{Ext}^i_A(M'/I^tM', N), \end{aligned} \tag{3.1}$$

and so the short exact sequence

$$0 \to Z \to M/I^{s+t}M \to M''/I^{s+t}M'' \to 0$$

leads to:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_A(M''/I^{s+t}M'', N) \arrow[r] & \mathrm{Hom}_A(M/I^{s+t}M, N) \arrow[r] & \mathrm{Hom}_A(Z, N) \arrow[dll, rounded corners] & \\
& \mathrm{Ext}^1_A(M''/I^{s+t}M'', N) \arrow[r] & \mathrm{Ext}^1_A(M/I^{s+t}M, N) \arrow[r] & \mathrm{Ext}^1_A(Z, N) \arrow[r] & \cdots
\end{tikzcd}$$

Taking the direct limit with respect to t of each term above, and applying (3.1) we are done. ■

Since for any nonzero finitely generated A-module N, $N/\mathfrak{m}N$ is a finitely generated $A/\mathfrak{m}$-vector space, it is not hard to show that one could take S_i to either be $\mathrm{Syz}_i(A/\mathfrak{m})$ or $\mathrm{Syz}_i(N/\mathfrak{m}N)$ for any nonzero finitely generated A-module N when studying the map $\mathrm{Ext}^i_A(M/\mathfrak{m}M, S_i) \to \mathrm{H}^i_{\mathfrak{m}}(M, S_i)$.

4 Main Results

4.1 Connections to the Canonical Element Conjecture

First we show a strong connection between the Canonical Element Conjecture and the maps $\operatorname{Ext}^i_A(M/\mathfrak{m}M, S_i) \to \mathrm{H}^i_{\mathfrak{m}}(M, S_i)$.

Proposition 4.1. *Let* $(A, \mathfrak{m})$ *be a local ring, then*

$$\operatorname{Ext}^i_A(A/\mathfrak{m}, \operatorname{Syz}_i(A/\mathfrak{m})) \to \mathrm{H}^i_{\mathfrak{m}}(\operatorname{Syz}_i(A/\mathfrak{m})) \qquad \textit{is nonzero}$$

if and only if for all nonzero finitely generated A*-modules* M

$$\operatorname{Ext}^i_A(M/\mathfrak{m}M, \operatorname{Syz}_i(A/\mathfrak{m})) \to \mathrm{H}^i_{\mathfrak{m}}(M, \operatorname{Syz}_i(A/\mathfrak{m})) \qquad \textit{is nonzero.}$$

Proof The reverse implication is clear. To see the forward implication, note that for any finitely generated A-module M we may write

$$A^r \to M \to 0,$$

where r is the number of generators in a minimal generating set of M. Applying $- \otimes_A A/\mathfrak{m}$, we see that $A^r/\mathfrak{m}A^r \simeq M/\mathfrak{m}M$. Thus we have the following commutative diagram:

$$\begin{tikzcd}
\operatorname{Ext}^i_A(M/\mathfrak{m}M, \operatorname{Syz}_i(A/\mathfrak{m})) \arrow[r, equal] \arrow[d] & \operatorname{Ext}^i_A(A^r/\mathfrak{m}A^r, \operatorname{Syz}_i(A/\mathfrak{m})) \arrow[d, "\neq 0"] \\
\mathrm{H}^i_{\mathfrak{m}}(M, \operatorname{Syz}_i(A/\mathfrak{m})) \arrow[r] & \mathrm{H}^i_{\mathfrak{m}}(A^r, \operatorname{Syz}_i(A/\mathfrak{m}))
\end{tikzcd}$$

By assumption

$$\operatorname{Ext}^i_A(A/\mathfrak{m}, \operatorname{Syz}_i(A/\mathfrak{m})) \to \mathrm{H}^i_{\mathfrak{m}}(\operatorname{Syz}_i(A/\mathfrak{m}))$$

is nonzero. From the definition of generalized local cohomology and the fact that Ext commutes with finite direct sums, we see that the right vertical map is nonzero. Hence the left vertical map is nonzero. ■

From Theorem 2.20, we see that given a local ring $(A, \mathfrak{m})$ of dimension n, the maps

$$\mathrm{Ext}^i_A(A/\mathfrak{m}, \mathrm{Syz}_i(A/\mathfrak{m})) \to \mathrm{H}^i_{\mathfrak{m}}(\mathrm{Syz}_i(A/\mathfrak{m}))$$

are nonzero for $0 \leqslant i \leqslant n-1$. Hence by Proposition 4.1, we see that for all finitely generated nonzero A-modules M the maps

$$\mathrm{Ext}^i_A(M/\mathfrak{m}M, \mathrm{Syz}_i(M/\mathfrak{m}M)) \to \mathrm{H}^i_{\mathfrak{m}}(M, \mathrm{Syz}_i(M/\mathfrak{m}M))$$

are nonzero for $0 \leqslant i \leqslant n-1$. We will now give a direct proof of this fact, using a line of reasoning similar to that found in [8, Theorem 3.2] and [11, Theorem 1.2]. The proof of this fact will illuminate connections that we will utilize later in this paper. We need the following proposition, [5, Proposition 1.1]:

Proposition 4.2 (Dutta [5, Proposition 1.1]). *Let A be a local ring and let*

$$G_\bullet : \qquad \cdots \to G_i \to \cdots \to G_1 \to G_0$$

be a complex of finitely generated free A-modules with $H_0(G_\bullet) = M$. Let N be a submodule of M. Then we can find a complex $L_\bullet$ of finitely generated free modules and a map $\psi_\bullet : L_\bullet \to G_\bullet$ such that:

(1) *$H_0(L_\bullet) = N$.*

(2) *$H_i(L_\bullet) \simeq H_i(G_\bullet)$ for all positive i.*

(3) *If $G_\bullet$ is minimal, then so is $L_\bullet$.*

(4) *$\mathrm{Cone}(\psi_\bullet)$ is a resolution of M/N.*

The above proposition is useful for constructing maps of complexes from a single map of modules, even when the target complex is not exact. This fact will be utilized in the theorem below.

Theorem 4.3. *Let $(A, \mathfrak{m})$ be a local ring of dimension n. For all nonzero finitely generated A-modules M the canonical map*

$$\mathrm{Ext}^i_A(M/\mathfrak{m}M, S_i) \to \mathrm{H}^i_{\mathfrak{m}}(M, S_i)$$

is nonzero for $0 \leqslant i \leqslant n-1$, where $S_i = \mathrm{Syz}_i(M/\mathfrak{m}M)$.

Proof Argue by way of contradiction. Suppose that for some integer $0 \leqslant i \leqslant n-1$, the map

$$\mathrm{Ext}^i_A(M/\mathfrak{m}M, S_i) \to \mathrm{H}^i_{\mathfrak{m}}(M, S_i)$$

is zero. Then for t sufficiently large, the canonical map

$$\mathrm{Ext}^i_A(M/\mathfrak{m}M, S_i) \to \mathrm{Ext}^i_A(M/\mathfrak{m}^t M, S_i)$$

is zero. Let

$$\cdots \longrightarrow F_i \xrightarrow{\sigma_i} F_{i-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\sigma_1} F_0 \longrightarrow M/\mathfrak{m}M \longrightarrow 0$$

and

$$\cdots \longrightarrow G_i \xrightarrow{\tau_i} G_{i-1} \longrightarrow \cdots \longrightarrow G_1 \xrightarrow{\tau_1} G_0 \longrightarrow M/\mathfrak{m}^tM \longrightarrow 0$$

be minimal free resolutions of $M/\mathfrak{m}M$ and $M/\mathfrak{m}^tM$ respectively. Consider the following commutative diagram, where $\varphi_\bullet$ is a lift of the canonical surjection $M/\mathfrak{m}^tM \to M/\mathfrak{m}M$, $T_i = \mathrm{Syz}_i(M/\mathfrak{m}^tM)$, and ι_T along with ι_S are canonical injections:

$$\begin{tikzcd}
G_i \arrow[rr, "\tau_i"] \arrow[dr] \arrow[dd, "\varphi_i"'] & & G_{i-1} \arrow[dd, "\varphi_{i-1}"] \arrow[dddl, dashed, "\delta"] \\
& T_i \arrow[ur, "\iota_T"'] \arrow[dd] & \\
F_i \arrow[rr, crossing over] \arrow[dr] & & F_{i-1} \\
& S_i \arrow[ur, "\iota_S"'] &
\end{tikzcd}$$

Set $(-)^\vee = \mathrm{Hom}_A(-, S_i)$ and note that

$$\begin{aligned}
\mathrm{Ext}^i_A(M/\mathfrak{m}^tM, S_i) &= \frac{\mathrm{Ker}(\tau_{i+1}^\vee)}{\mathrm{Im}(\tau_i^\vee)} \\
&\simeq \frac{\mathrm{Hom}_A(T_i, S_i)}{\mathrm{Im}(\tau_i^\vee)}.
\end{aligned}$$

By assumption, the class of φ_i is zero in $\mathrm{Ext}^i_A(M/\mathfrak{m}^tM, S_i)$. Thus the image of $\varphi_{i-1}|_{T_i}$ is zero in $\mathrm{Ext}^i_A(M/\mathfrak{m}^tM, S_i)$, by the commutativity of the diagram above. Hence the image of the class of $\varphi_{i-1}|_{T_i}$ is in $\mathrm{Im}(\tau_i^\vee)$. So there exists

$$\delta : G_{i-1} \to S_i$$

such that $\delta \circ \iota_T = \varphi_{i-1}|_{T_i}$. Now set

$$\begin{aligned}
\widetilde{\varphi}_i &:= \varphi_{i-1}|_{T_i} - \delta \circ \iota_T, \\
\widetilde{\varphi}_{i-1} &:= \varphi_{i-1} - \iota_S \circ \delta.
\end{aligned}$$

Noting $\widetilde{\varphi}_i = 0$ on T_i, we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & T_i \arrow[r] \arrow[d, "0"] & G_{i-1} \arrow[r] \arrow[d, "\widetilde{\varphi}_{i-1}"] & T_{i-1} \arrow[r] \arrow[d, "\varphi_{i-2}"] & 0 \\
0 \arrow[r] & S_i \arrow[r] & F_{i-1} \arrow[r] & T_{i-1} \arrow[r] & 0
\end{tikzcd}$$

If we continue to lift the map $\widetilde{\varphi}_{i-1}$, the vertical maps of higher degree can be taken to all be zero. Thus we have the following situation:

$$\begin{tikzcd}
\cdots \arrow[r] & G_i \arrow[r, "\tau_i"] \arrow[d, "0"] & G_{i-1} \arrow[r] \arrow[d, "\varphi_{i-1}"] & \cdots \arrow[r] & G_1 \arrow[r] \arrow[d, "\varphi_1"] & M/\mathfrak{m}^t M \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & F_i \arrow[r, "\sigma_i"] & F_{i-1} \arrow[r] & \cdots \arrow[r] & F_1 \arrow[r] & M/\mathfrak{m}M \arrow[r] & 0
\end{tikzcd}$$

Apply $(-)^* = \operatorname{Hom}_A(-, A)$ to the diagram above to obtain:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ext}^{i-1}_A(M/\mathfrak{m}M, A) \arrow[r] \arrow[d] & F^*_{i-1}/\operatorname{Im}(\sigma^*_{i-1}) \arrow[r] \arrow[d, "\widetilde{\varphi}^*_{i-1}"] \arrow[dl, dashed, "\ell"] & \operatorname{Im}(\sigma^*_i) \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & \operatorname{Ext}^{i-1}_A(M/\mathfrak{m}^t M, A) \arrow[r] & G^*_{i-1}/\operatorname{Im}(\tau^*_{i-1}) \arrow[r] & \operatorname{Im}(\tau^*_i) \arrow[r] & 0
\end{tikzcd}$$

Since the far right vertical map is 0, we obtain a lift ℓ making the above diagram commute. Thus we see that the composition

$$F^*_{i-1}/\operatorname{Im}(\sigma^*_{i-1}) \to \operatorname{Ext}^{i-1}_A(M/\mathfrak{m}^t M, A) \to G^*_{i-1}/\operatorname{Im}(\tau^*_{i-1}) \to \operatorname{Im}(\tau^*_i) \tag{4.1}$$

is the zero map.

From Proposition 4.2, [5, Proposition 1.1], we obtain a minimal complex $L_\bullet$ such that the following diagram commutes:

$$\begin{tikzcd}
L_i \arrow[r, "\lambda_i"] \arrow[d] & L_{i-1} \arrow[r, "\lambda_{i-1}"] \arrow[d, "\psi_{i-1}"] & L_{i-2} \arrow[r] \arrow[d, "\psi_{i-2}"] & \cdots \arrow[r] & L_0 \arrow[r] \arrow[d, "\psi_0"] & \operatorname{Ext}^{i-1}_A(M/\mathfrak{m}^t M, A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] \arrow[d] & G^*_0 \arrow[r] \arrow[d] & G^*_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & G^*_{i-1} \arrow[r] \arrow[d] & G^*_{i-1}/\operatorname{Im}(\tau^*_{i-1}) \arrow[r] \arrow[d] & 0 \\
L_{i-1} \arrow[r] & L_{i-2} \oplus G^*_0 \arrow[r] & L_{i-3} \oplus G^*_1 \arrow[r] & \cdots \arrow[r] & G^*_{i-1} \arrow[r] & \operatorname{Im}(\tau^*_i) \arrow[r] & 0
\end{tikzcd} \tag{4.2}$$

Note that the bottom row is the cone of $\psi_\bullet$, which again by Proposition 4.2, is a minimal free resolution of $\operatorname{Im}(\tau^*_i)$. In particular, note that by the definition of the mapping cone we have:

$$\begin{aligned}
L_{i-1} &\to L_{i-2} \oplus G^*_0 \\
l &\mapsto (-\lambda_{i-1}(l), -\psi_{i-1}(l))
\end{aligned}$$

Now consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & F_0^* \arrow[r, "\sigma_0^*"] \arrow[d, equal] \arrow[ddl, dashed, "h"'] & F_1^* \arrow[r] \arrow[d] \arrow[ddl, dashed, "h'"'] & \cdots \arrow[r] & F_{i-1}^* \arrow[r] \arrow[d] & F_{i-1}^*/\operatorname{Im}(\sigma_{i-1}^*) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] \arrow[d] & G_0^* \arrow[r] \arrow[d, "{(0,\mathbb{1}_{G_0^*})}"'] & G_1^* \arrow[r] \arrow[d] & \cdots \arrow[r] & G_{i-1}^* \arrow[r] \arrow[d] & G_{i-1}^*/\operatorname{Im}(\tau_{i-1}^*) \arrow[r] \arrow[d] & 0 \\
L_{i-1} \arrow[r] & L_{i-2} \oplus G_0^* \arrow[r] & L_{i-3} \oplus G_1^* \arrow[r] & \cdots \arrow[r] & G_{i-1}^* \arrow[r] & \operatorname{Im}(\tau_i^*) \arrow[r] & 0
\end{tikzcd}$$

By the composition given in line (4.1), the lift given above is homotopic to zero. Thus there exist homotopy maps h, h' such that:

$$(-\lambda_{i-1}, -\psi_{i-1}) \circ h + h' \circ \sigma_0^* = (0, \mathbb{1}_{G_0^*})$$

However, $\operatorname{Im}(\sigma_0^*) \subset \mathfrak{m}F_1^*$, thus we see:

$$\operatorname{Im}(\psi_{i-1}) = \operatorname{Im}(-\psi_{i-1}) = G_0^*$$

This together with diagram (4.2), shows that there are minimal generators of L_{i-1} which are not in $\operatorname{Im}(\lambda_i)$, namely the ones that map onto the minimal generators of G_0^*. Thus

$$\operatorname{Coker}(\lambda_i) = \frac{L_{i-1}}{\operatorname{Im}(\lambda_i)}$$

has a free summand. By construction, the homology of $L_\bullet$ is:

$$H_j(L_\bullet) = \begin{cases} \operatorname{Ext}_A^{i-j-1}(M/\mathfrak{m}^t M, A) & \text{for } j < i, \\ 0 & \text{for } j \geqslant i \end{cases}$$

Note that $\ell(H_j(L_\bullet)) < \infty$ for all j. Moreover, if $d = \operatorname{depth}(A)$, for $j \geqslant i - d$, $H_j(L_\bullet) = 0$. Thus for $0 \leqslant i \leqslant n-1$, $L_\bullet$ satisfies the hypothesis of Theorem 1.1, [6, Theorem 1.1], and the cokernel of λ_i has a free summand, a contradiction. ■

While the validity of [8, Theorem 3.2] and [11, Theorem 1.2] both rely on the validity of the Improved New Intersection Conjecture in equicharacteristic, the argument above relies on Theorem 1.1, [6, Theorem 1.1]. This observation is interesting since the Improved New Intersection Conjecture is a statement about complexes that seems meaningful only in degree less than the dimension of the ring, while Theorem 1.1 is a statement that is meaningful in any degree. Despite this seeming advantage, it seems difficult to use it to obtain a stronger result.

4.2 Infinite Projective Dimension

In the case of standard local cohomology, the local cohomology modules all vanish above the dimension of the ring. This is not necessarily the case for generalized local cohomology. Thus given a local ring $(A, \mathfrak{m})$ and a finitely generated nonzero A-module M, we may construct the following commutative diagram:

$$\begin{tikzcd}
E^0 \arrow[r, "\delta_0"] \arrow[d, "\vartheta_0"] & E^1 \arrow[r, "\delta_1"] \arrow[d, "\vartheta_1"] & \cdots \arrow[r, "\delta_{i-2}"] & E^{i-1} \arrow[r, "\delta_{i-1}"] \arrow[d, "\vartheta_{i-1}"] & E^i \arrow[r, "\delta_i"] \arrow[d, "\vartheta_i"] & \cdots \\
H^0 \arrow[r, "\overline{\delta}_0"] & H^1 \arrow[r, "\overline{\delta}_1"] & \cdots \arrow[r, "\overline{\delta}_{i-2}"] & H^{i-1} \arrow[r, "\overline{\delta}_{i-1}"] & H^i \arrow[r, "\overline{\delta}_i"] & \cdots
\end{tikzcd}$$

Here $E^i = \operatorname{Ext}^i_A(M/\mathfrak{m}M, S_i)$, $H^i = \operatorname{H}^i_{\mathfrak{m}}(M, S_i)$, $S_i = \operatorname{Syz}_i(M/\mathfrak{m}M)$, and the vertical maps are from the definition of $\operatorname{H}^i_{\mathfrak{m}}(M, -)$. Since E^0 is an $A/\mathfrak{m}$ vector space and its generators are preserved by the δ_i's, the commutativity of the diagram shows that if ϑ_i is nonzero, then ϑ_j is nonzero for $0 \leqslant j \leqslant i$. Our next theorem gives a construction for modules such that ϑ_i is nonzero, even when the validity of the Canonical Element Conjecture is unknown.

Theorem 4.4. *Let $(A, \mathfrak{m})$ be a local ring which is not regular and let $(P_\bullet, \rho_\bullet)$ be a minimal free resolution of some finitely generated A-module Q of infinite projective dimension. If $M = \operatorname{Coker}(\rho_i^*)$ for some $i > 2$, then the canonical map*

$$\operatorname{Ext}^i_A(M/\mathfrak{m}M, S_i) \to \operatorname{H}^i_{\mathfrak{m}}(M, S_i)$$

is nonzero, where $S_i = \operatorname{Syz}_i(M/\mathfrak{m}M)$.

Proof Argue by way of contradiction. Suppose that for some natural number i, the map

$$\operatorname{Ext}^i_A(M/\mathfrak{m}M, S_i) \to \operatorname{H}^i_{\mathfrak{m}}(M, S_i)$$

is zero. Then for t sufficiently large, the canonical map

$$\operatorname{Ext}^i_A(M/\mathfrak{m}M, S_i) \to \operatorname{Ext}^i_A(M/\mathfrak{m}^tM, S_i)$$

is zero. Let

$$\cdots \longrightarrow F_i \xrightarrow{\sigma_i} F_{i-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\sigma_1} F_0 \longrightarrow M/\mathfrak{m}M \longrightarrow 0$$

and

$$\cdots \longrightarrow G_i \xrightarrow{\tau_i} G_{i-1} \longrightarrow \cdots \longrightarrow G_1 \xrightarrow{\tau_1} G_0 \longrightarrow M/\mathfrak{m}^tM \longrightarrow 0$$

be minimal free resolutions of $M/\mathfrak{m}M$ and $M/\mathfrak{m}^tM$ respectively. Consider the following commutative diagram, where $\varphi_\bullet$ is a lift of the canonical surjection $M/\mathfrak{m}^tM \to M/\mathfrak{m}M$, $T_i = \operatorname{Syz}_i(M/\mathfrak{m}^tM)$, and ι_T along with ι_S are canonical injections:

$$\begin{tikzcd}
G_i \arrow[rr, "\tau_i"] \arrow[dr] \arrow[dd, "\varphi_i"'] & & G_{i-1} \arrow[dd, "\varphi_{i-1}"] \arrow[dddl, dashed, "\delta"] \\
& T_i \arrow[ur, "\iota_T"] \arrow[dd] & \\
F_i \arrow[rr] \arrow[dr] & & F_{i-1} \\
& S_i \arrow[ur, "\iota_S"'] &
\end{tikzcd}$$

Set $(-)^\vee = \operatorname{Hom}_A(-, S_i)$ and note that

$$\begin{aligned}
\operatorname{Ext}^i_A(M/\mathfrak{m}^t M, S_i) &= \frac{\operatorname{Ker}(\tau_{i+1}^\vee)}{\operatorname{Im}(\tau_i^\vee)} \\
&\simeq \frac{\operatorname{Hom}_A(T_i, S_i)}{\operatorname{Im}(\tau_i^\vee)}.
\end{aligned}$$

By assumption, the class of φ_i is 0 in $\operatorname{Ext}^i_A(M/\mathfrak{m}^t M, S_i)$. By the same argument used in the proof of Theorem 4.3, we obtain the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & T_i \arrow[r] \arrow[d, "0"] & G_{i-1} \arrow[r] \arrow[d, "\widetilde{\varphi}_{i-1}"] & T_{i-1} \arrow[r] \arrow[d, "\varphi_{i-2}"] & 0 \\
0 \arrow[r] & S_i \arrow[r] & F_{i-1} \arrow[r] & S_{i-1} \arrow[r] & 0
\end{tikzcd}$$

Thus WLOG we may assume that φ_i is the zero map. Now apply the functor $(-)^* = \operatorname{Hom}_A(-, A)$ and write:

$$\begin{tikzcd}
0 \arrow[rr] & & \operatorname{Ker}(\sigma_i^*) \arrow[r] \arrow[dd] & F_{i-1}^* \arrow[r] \arrow[ddl, "\varepsilon"'] \arrow[dd, "\varphi_{i-1}^*"] & \operatorname{Im}(\sigma_i^*) \arrow[r] \arrow[dd, "0"] & 0 \\
& \operatorname{Im}(\sigma_{i-1}^*) \arrow[ur] \arrow[dd, "\varphi^*"] & & & & \\
0 \arrow[rr] & & \operatorname{Ker}(\tau_i^*) \arrow[r] & G_{i-1}^* \arrow[r] & \operatorname{Im}(\tau_i^*) \arrow[r] & 0 \\
& \operatorname{Im}(\tau_{i-1}^*) \arrow[ur, "\psi"'] & & & &
\end{tikzcd}$$

Here $\varepsilon : F_{i-1}^* \to \operatorname{Ker}(\tau_i^*)$ is a homomorphism obtained from the right hand square of the commutative diagram above. Setting $\varphi^* = \varphi_{i-1}^*|_{\operatorname{Im}(\sigma_{i-1}^*)}$ and letting $\iota_F : \operatorname{Im}(\sigma_{i-1}^*) \to F_{i-1}^*$ and ψ be the canonical injections, we have that:

$$\psi \circ \varphi^* = \varepsilon \circ \iota_F \tag{4.3}$$

Let

$$\cdots \longrightarrow L_i \xrightarrow{\lambda_i} L_{i-1} \longrightarrow \cdots \longrightarrow L_1 \xrightarrow{\lambda_1} L_0 \longrightarrow \operatorname{Ker}(\tau_i^*) \longrightarrow 0$$

be a minimal free resolution of $\operatorname{Ker}(\tau_i^*)$ and write:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & F_0^* \arrow[r, "\sigma_1^*"] \arrow[d, equal, "\varphi_0^*"] & F_1^* \arrow[r] \arrow[d, "\varphi_1^*"] & \cdots \arrow[r] & F_{i-2}^* \arrow[r, two heads] \arrow[d, "\varphi_{i-2}^*"] & \operatorname{Im}(\sigma_{i-1}^*) \arrow[d, "\varphi^*"] & \\
0 \arrow[r] \arrow[d] & G_0^* \arrow[r, "\tau_1^*"] \arrow[d, "\psi_{i-2}"] & G_1^* \arrow[r] \arrow[d, "\psi_{i-3}"] & \cdots \arrow[r] & G_{i-2}^* \arrow[r, two heads] \arrow[d, "\psi_0"] & \operatorname{Im}(\tau_{i-1}^*) \arrow[d, "\psi"] & \\
L_{i-1} \arrow[r, "\lambda_{i-1}"] & L_{i-2} \arrow[r, "\lambda_{i-2}"] & L_{i-3} \arrow[r] & \cdots \arrow[r] & L_0 \arrow[r] & \operatorname{Ker}(\tau_i^*) \arrow[r] & 0
\end{tikzcd}$$

However, we also have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & F_0^* \arrow[r, "\sigma_1^*"] \arrow[d] & F_1^* \arrow[r] \arrow[d] & \cdots \arrow[r] & F_{i-3}^* \arrow[r, "\sigma_{i-2}^*"] \arrow[d] & F_{i-2}^* \arrow[r, two heads] \arrow[d, "\sigma_{i-1}^*"] & \operatorname{Im}(\sigma_{i-1}^*) \arrow[d, "\iota_F"] & \\
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & F_{i-1}^* \arrow[r] \arrow[d] & F_{i-1}^* \arrow[r] \arrow[d, "\varepsilon"] & 0 \\
L_{i-1} \arrow[r, "\lambda_{i-1}"] & L_{i-2} \arrow[r, "\lambda_{i-2}"] & L_{i-3} \arrow[r] & \cdots \arrow[r] & L_1 \arrow[r, "\lambda_1"] & L_0 \arrow[r] & \operatorname{Ker}(\tau_i^*) \arrow[r] & 0
\end{tikzcd}$$

By (4.3) above we see that we have two lifts of the map $\psi \circ \varphi^* = \varepsilon \circ \iota_F$. Hence we have homotopy maps g and g' such that

$$\begin{aligned}
\psi_{i-2} \circ \varphi_0^* &= \lambda_{i-1} \circ g + g' \circ \sigma_1^*, && \text{hence} \\
\operatorname{Im}(\psi_{i-2} \circ \varphi_0^*) &= \operatorname{Im}(\lambda_{i-1} \circ g + g' \circ \sigma_1^*),
\end{aligned}$$

with right hand side of the above equation contained in $\mathfrak{m}L_{i-2}$ as the maps λ_{i-1} and σ_1^* are minimal. This leads to a contradiction as we will now show that:

$$\operatorname{Im}(\psi_{i-2} \circ \varphi_0^*) \not\subset \mathfrak{m}L_{i-2}$$

Recall the construction of M and write:

$$\begin{tikzcd}
P_0^* \arrow[r] \arrow[d, "\theta_i"] & P_1^* \arrow[r] \arrow[d, "\theta_{i-1}"] & P_2^* \arrow[r] \arrow[d, "\theta_{i-2}"] & \cdots \arrow[r] & P_{i-1}^* \arrow[r] \arrow[d, "\theta_1"] & P_i^* \arrow[r] \arrow[d, equal, "\theta_0"] & M \arrow[r] \arrow[d, "\theta"] & 0 \\
G_i \arrow[r] & G_{i-1} \arrow[r] & G_{i-2} \arrow[r] & \cdots \arrow[r] & G_1 \arrow[r] & G_0 \arrow[r] & M/\mathfrak{m}^t M \arrow[r] & 0
\end{tikzcd}$$

Apply $(-)^*$ and look at:

$$\begin{tikzcd}
G_{i-2}^* \arrow[rrr, "\tau_{i-1}^*"] \arrow[dr] \arrow[dd] & & & G_{i-1}^* \arrow[r, "\tau_i^*"] \arrow[dd, "\theta_{i-1}^*"] & G_i^* \arrow[dd] \\
& \operatorname{Im}(\tau_{i-1}^*) \arrow[r, hook, "\psi"] & \operatorname{Ker}(\tau_i^*) \arrow[ur] \arrow[dd, "\gamma"] & & \\
P_2 \arrow[rrr] \arrow[drr] & & & P_1 \arrow[r] & P_0 \\
& & U \arrow[ur] & &
\end{tikzcd}$$

where $U = \operatorname{Syz}_2(Q)$ and $\gamma = \theta_{i-1}^*|_{\operatorname{Ker}(\tau_i^*)}$. Letting $\widetilde{\theta} = \gamma \circ \psi$, we have the following commutative triangle:

$$\begin{tikzcd}
\operatorname{Im}(\tau_{i-1}^*) \arrow[dr, "\psi"] \arrow[dd, "\widetilde{\theta}"] & \\
& \operatorname{Ker}(\tau_i^*) \arrow[dl, "\gamma"] \\
U &
\end{tikzcd}$$

From our work above, we have three complexes $G_\bullet^* \to \operatorname{Im}(\tau_{i-1}^*)$, $L_\bullet \to \operatorname{Ker}(\tau_i^*)$, and $P_\bullet \to U$. Let $\psi_\bullet$ be a lift of ψ, $\gamma_\bullet$ be a lift of γ, and note that $\theta_\bullet^*$ lifts $\widetilde{\theta}$. We may put these lifts and complexes together into a long commutative diagram which we examine near degree $i-2$:

$$\begin{tikzcd}
0 \arrow[rr] \arrow[dr] \arrow[dd] & & G_0^* \arrow[rr, "\tau_1^*"] \arrow[dr, "\psi_{i-2}"] \arrow[dd, equal, "\theta_0^*"'] & & G_1^* \arrow[rr] \arrow[dr] \arrow[dd] & & \cdots & \\
& L_{i-1} \arrow[rr] \arrow[dl] & & L_{i-2} \arrow[rr] \arrow[dl, "\gamma_{i-2}"] & & L_{i-3} \arrow[rr] \arrow[dl] & & \cdots \\
P_{i+1} \arrow[rr, "\rho_{i+1}"] & & P_i \arrow[rr] & & P_{i-1} \arrow[rr] & & \cdots &
\end{tikzcd}$$

Since $\theta_\bullet^*$ and $\gamma_\bullet \circ \psi_\bullet$ are both lifts of $\widetilde{\theta} = \gamma \circ \psi$, we see that there are homotopy maps, h and h', such that:

$$\theta_0^* - \gamma_{i-2} \circ \psi_{i-2} = \rho_{i+1} \circ h + h' \circ \tau_1^*$$

However, $\operatorname{Im}(\rho_{i+1} \circ h + h' \circ \tau_1^*) \subset \mathfrak{m}P_i$, and so

$$(\theta_0^* - \gamma_{i-2} \circ \psi_{i-2}) \otimes_A A/\mathfrak{m} = 0.$$

Hence θ_0^* and $\gamma_{i-2} \circ \psi_{i-2}$ agree on minimal generators modulo $\mathfrak{m}$. Thus we see that $\operatorname{Im}(\psi_{i-2}) \not\subset \mathfrak{m}L_{i-2}$ and so $\operatorname{Im}(\psi_{i-2} \circ \varphi_0^*) \not\subset \mathfrak{m}L_{i-2}$, a contradiction. ■

By the discussion given at the beginning of this section, we see that if we construct an A-module M such that for some $i > 2$

$$\operatorname{Ext}_A^i(M/\mathfrak{m}M, \operatorname{Syz}_i(M/\mathfrak{m}M)) \to \operatorname{H}_{\mathfrak{m}}^i(M, \operatorname{Syz}_i(M/\mathfrak{m}M))$$

is nonzero, then

$$\operatorname{Ext}^j_A(M/\mathfrak{m}M, \operatorname{Syz}_j(M/\mathfrak{m}M)) \to \operatorname{H}^j_{\mathfrak{m}}(M, \operatorname{Syz}_j(M/\mathfrak{m}M))$$

is nonzero for all $0 \leqslant j \leqslant i$. One may wonder if there are rings and modules such that the map

$$\operatorname{Ext}^i_A(M/\mathfrak{m}M, \operatorname{Syz}_i(M/\mathfrak{m}M)) \to \operatorname{H}^i_{\mathfrak{m}}(M, \operatorname{Syz}_i(M/\mathfrak{m}M))$$

is nonzero for all $i \geqslant 0$. Our next corollary answers this question in the affirmative:

Corollary 4.5. *Let $(A, \mathfrak{m})$ be a nonregular Gorenstein ring of dimension n. Then there exist infinitely many isomorphism classes of finitely generated A-modules M such that the canonical map*

$$\operatorname{Ext}^i_A(M/\mathfrak{m}M, \operatorname{Syz}_i(M/\mathfrak{m}M)) \to \operatorname{H}^i_{\mathfrak{m}}(M, \operatorname{Syz}_i(M/\mathfrak{m}M))$$

is nonzero for all $i \geqslant 0$.

Proof Consider a module M constructed as in Theorem 4.4 where $i = n+1$. Let $F_\bullet$ be a minimal free resolution of $M/\mathfrak{m}M$ and let $S_i = \operatorname{Syz}_i(M/\mathfrak{m}M)$. Write

$$0 \longrightarrow S_{n+2} \longrightarrow F_{n+1} \longrightarrow S_{n+1} \longrightarrow 0$$

and look at the corresponding long exact sequences of Ext and generalized local cohomology modules:

$$\begin{tikzcd}
 & \operatorname{Ext}^{n+1}_A(M/\mathfrak{m}M, S_{n+1}) \arrow[r] \arrow[d, "\neq 0"] & \operatorname{Ext}^{n+2}_A(M/\mathfrak{m}M, S_{n+2}) \arrow[d] \\
\operatorname{H}^{n+1}_{\mathfrak{m}}(M, F_{n+1}) \arrow[r] & \operatorname{H}^{n+1}_{\mathfrak{m}}(M, S_{n+1}) \arrow[r, "\delta"] & \operatorname{H}^{n+2}_{\mathfrak{m}}(M, S_{n+2})
\end{tikzcd}$$

Since A is Gorenstein, A has injective dimension n or less. Thus

$$\operatorname{H}^{n+1}_{\mathfrak{m}}(M, F_{n+1}) = \varinjlim \operatorname{Ext}^{n+1}_A(M/\mathfrak{m}^t M, F_{n+1}) = 0.$$

Thus δ above is injective, and so

$$\operatorname{Ext}^{n+2}_A(M/\mathfrak{m}M, S_{n+2}) \xrightarrow{\neq 0} \operatorname{H}^{n+2}_{\mathfrak{m}}(M, S_{n+2}).$$

A similar proof will work for any $i > n$. ■

Remark. It would be interesting to know if there are non-Gorenstein rings such that $\operatorname{H}^i_{\mathfrak{m}}(M, A) = 0$ for $i \gg 0$, where M has infinite projective dimension.

4.3 Cokernels and Free Summands

Using a technique similar to the one used in the proof of Theorem 4.4, we can now construct infinitely many isomorphism classes of complexes $L_\bullet$, as described in the hypothesis of Theorem 1.1, such that $\mathrm{Coker}(\lambda_n)$ cannot have a free summand.

Theorem 4.6. *Let $(A, \mathfrak{m})$ be a local ring of dimension n and depth d such that $n - d > 1$. Then there exists infinitely many isomorphism classes of complexes of finitely generated free modules*

$$L_\bullet : \qquad \cdots \longrightarrow L_i \xrightarrow{\lambda_i} L_{i-1} \longrightarrow \cdots \longrightarrow L_1 \xrightarrow{\lambda_1} L_0 \longrightarrow 0$$

such that:

(1) $H_0(L_\bullet) \neq 0$.

(2) $\ell(H_i(L_\bullet)) < \infty$.

(3) $H_i(L_\bullet) = 0$ *for* $i \geqslant n - d$.

(4) $\mathrm{Coker}(\lambda_n)$ *does not have a free summand.*

Proof For $n \leqslant 3$, the result follows from Theorem 1.1 and the work of Heitmann [16]. Assume that $n > 3$ and let

$$P_\bullet : \qquad \cdots \longrightarrow P_i \xrightarrow{\rho_i} P_{i-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{\rho_1} P_0 \longrightarrow Q \longrightarrow 0$$

be a minimal free resolution of any finitely generated A-module Q of infinite projective dimension. Let $M = \mathrm{Coker}(\rho_n^*)$, where $(-)^* = \mathrm{Hom}_A(-, A)$ and let

$$F_\bullet : \qquad \cdots \longrightarrow F_i \xrightarrow{\sigma_i} F_{i-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\sigma_1} F_0 \longrightarrow M/\mathfrak{m}M \longrightarrow 0$$

be a minimal free resolution of M. Write:

$$\begin{tikzcd}
P_0^* \arrow[r] \arrow[d, "\theta_n"] & P_1^* \arrow[r] \arrow[d, "\theta_{n-1}"] & P_2^* \arrow[r] \arrow[d, "\theta_{n-2}"] & \cdots \arrow[r] & P_{n-1}^* \arrow[r] \arrow[d, "\theta_1"] & P_n^* \arrow[r] \arrow[d, equal, "\theta_0"] & M \arrow[r] \arrow[d, "\theta"] & 0 \\
F_n \arrow[r] & F_{n-1} \arrow[r] & F_{n-2} \arrow[r] & \cdots \arrow[r] & F_1 \arrow[r] & F_0 \arrow[r] & M/\mathfrak{m}M \arrow[r] & 0
\end{tikzcd}$$

Working as in the proof of Theorem 4.4, set $U = \mathrm{Syz}_2(Q)$, $\gamma = \theta_{n-1}^*|_{\mathrm{Ker}(\sigma_n^*)}$, and let $\widetilde{\theta} = \gamma \circ \psi$. We have the following commutative triangle:

$$\begin{tikzcd}
\mathrm{Im}(\sigma_{n-1}^*) \arrow[dr, "\psi"] \arrow[dd, "\widetilde{\theta}"] & \\
& \mathrm{Ker}(\sigma_n^*) \arrow[dl, "\gamma"] \\
U &
\end{tikzcd}$$

Let $(G_\bullet, \tau_\bullet)$ be a minimal free resolution of $\operatorname{Ker}(\sigma_n^*)$. Let $\psi_\bullet$ be a lift of ψ, $\gamma_\bullet$ be a lift of γ, and note that $\theta_\bullet^*$ lifts $\widetilde{\theta}$. We may put these lifts and complexes together into a long commutative diagram which we examine near degree $n-2$:

$$\begin{tikzcd}
0 \arrow[rr] \arrow[dd] \arrow[dr, "\psi_{n-1}"] & & F_0^* \arrow[rr] \arrow[dd, equal, "\theta_0^*"'] \arrow[dr, "\psi_{n-2}"] & & F_1^* \arrow[rr] \arrow[dd, "\theta_1^*"'] \arrow[dr, "\psi_{n-3}"] & & \cdots & \\
& G_{n-1} \arrow[rr] \arrow[dl, "\gamma_{n-1}"] & & G_{n-2} \arrow[rr] \arrow[dl, "\gamma_{n-2}"] & & G_{n-3} \arrow[rr] \arrow[dl, "\gamma_{n-3}"] & & \cdots \\
P_{n+1} \arrow[rr] & & P_n \arrow[rr] & & P_{n-1} \arrow[rr] & & \cdots &
\end{tikzcd}$$

Now look at the mapping cone of $\psi_\bullet$:

$$\cdots \to G_n \to F_0^* \oplus G_{n-1} \to F_1^* \oplus G_{n-2} \to \cdots \to F_{n-2}^* \oplus G_1 \to G_0$$

Let $(L_\bullet, \lambda_\bullet)$ be the minimal subcomplex of the mapping cone of $\psi_\bullet$. Note that since

$$\gamma_{n-2} \circ \psi_{n-2} = \theta_0^*$$

is the identity homomorphism, $\psi_{n-2} : F_0^* \hookrightarrow G_{n-2}$ is an injection. Hence, $\operatorname{Coker}(\lambda_n)$ has a free summand if and only if $\operatorname{Coker}(\tau_n)$ has a free summand.

To show that $\operatorname{Coker}(\tau_n)$ cannot have a free summand, we will use the same technique as used in the proof of [6, Theorem 1.1]. Suppose that $\operatorname{Coker}(\tau_n)$ has a free summand. Write $\operatorname{Coker}(\tau_n) = A \oplus B$ and consider the following exact sequence:

$$0 \to \operatorname{Coker}(\tau_n) \to G_{n-2} \to \operatorname{Im}(\tau_{n-2}) \to 0$$

From this we obtain two more exact sequences:

$$0 \to A \to G_{n-2} \to V \to 0$$
$$0 \to B \to G_{n-2} \to W \to 0$$

Here V and W are the cokernels of the maps $A \to G_{n-2}$ and $B \to G_{n-2}$ respectively. Since $\operatorname{Coker}(\tau_n) = A \oplus B$ we obtain the short exact sequence:

$$0 \to G_{n-2} \to V \oplus W \to \operatorname{Im}(\tau_{n-2}) \to 0 \tag{4.4}$$

However, by repeatedly applying Proposition 2.10, $\operatorname{depth}(\operatorname{Im}(\tau_{n-2})) \geqslant n-2$ and so we conclude that $\operatorname{depth}(\operatorname{Im}(\tau_{n-2})) = d$. Since the projective dimension of V is 1, it has depth $d-1$. This is a contradiction, as by (4.4) above, $\operatorname{depth}(V \oplus W)$ must be at least d. Hence $\operatorname{Coker}(\tau_n)$ does not have a free summand and therefore neither does $\operatorname{Coker}(\lambda_n)$.

Now we'll examine the homology of $L_\bullet$. Since $L_\bullet$ is a minimal subcomplex of Cone($\psi_\bullet$), we have

$$H_i(L_\bullet) = \operatorname{Ext}_A^{n-i-1}(M/\mathfrak{m}M, A)$$

for $i \geqslant 0$. To see this when $i > 1$, consider the long exact sequence associated to the mapping cone. To see this when $0 \leqslant i \leqslant 1$, consider the following exact sequences:

$$0 \to H_1(L_\bullet) \to \frac{F_{n-2}^*}{\operatorname{Im}(\sigma_{n-2}^*)} \to \operatorname{Im}(\sigma_{n-1}^*) \to 0$$

and

$$0 \to \operatorname{Im}(\sigma_{n-1}^*) \to \operatorname{Ker}(\sigma_n^*) \to H_0(L_\bullet) \to 0$$

Thus:

(1) $H_0(L_\bullet) = \operatorname{Ext}_A^{n-1}(M/\mathfrak{m}M, A) \neq 0$.

(2) $\ell(H_i(L_\bullet)) < \infty$.

(3) $H_i(L_\bullet) = 0$ for $i \geqslant n - d$.

(4) Coker(λ_n) does not have a free summand.

Thus $L_\bullet$ is a complex exhibiting the desired properties. ■

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